

# COVERS, ENVELOPES, AND COTORSION THEORIES IN LOCALLY PRESENTABLE ABELIAN CATEGORIES AND CONTRAMODULE CATEGORIES

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**ABSTRACT.** We prove general results about completeness of cotorsion theories and existence of covers and envelopes in locally presentable abelian categories, extending the well-established theory for module categories and Grothendieck categories. These results are then applied to the categories of contramodules over topological rings, which provide examples and counterexamples.

## 1. INTRODUCTION

1.1. An abelian category with enough projective objects is determined by (can be recovered from) its full subcategory of projective objects. An explicit construction providing the recovery procedure is described, in the nonadditive context, in the paper [13] (see also [21]), where it is called “the exact completion of a weakly left exact category”, and in the additive context, in the paper [22, Lemma 1], where it is called “the category of coherent functors”. In fact, the exact completion in the additive context is contained already in [19], as it is explained in [35].

Hence, in particular, any abelian category with arbitrary coproducts and a single small (finitely generated, finitely presented) projective generator  $P$  is equivalent to the category of right modules over the ring of endomorphisms of  $P$ . More generally, an abelian category with arbitrary coproducts and a set of small projective generators is equivalent to the category of contravariant additive functors from its full subcategory formed by these generators into the category of abelian groups (right modules over the big ring of morphisms between the generators).

Now let  $\mathcal{K}$  be an abelian category with a single, not necessarily small projective generator  $P$ ; suppose that all coproducts of copies of  $P$  exists in  $\mathcal{K}$ . Then, for the same reasons, the category  $\mathcal{K}$  is determined by the functor  $\mathbb{T}$  assigning to a set  $S$  the set  $\mathcal{K}(P, P^{(S)})$  of all morphisms from  $P$  to the  $S$ -indexed coproduct of copies of  $P$ . The functor  $\mathbb{T}$  is endowed with a natural structure of a monad on the category of sets; the full subcategory of coproducts of copies of  $P$  in  $\mathcal{K}$  is the category of free algebras over this monad; and the whole category  $\mathcal{K}$  is equivalent to the category of

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all algebras over the monad  $\mathbb{T}$  [42]. The functor  $\mathcal{K}(P, -)$  assigns to an object of  $\mathcal{K}$  the underlying set of the corresponding  $\mathbb{T}$ -algebra.

Conversely, the category of algebras over a monad  $\mathbb{T}$  on the category of sets is abelian whenever it is additive. Given a monad  $\mathbb{T} : \mathbf{Set} \rightarrow \mathbf{Set}$ , elements of the set  $\mathbb{T}(S)$  can be viewed as  $S$ -ary operations on the underlying sets of algebras over  $\mathbb{T}$ , as the datum of an element  $t \in \mathbb{T}(S)$  allows to assign to any map of sets  $f : S \rightarrow A$  from  $S$  into a  $\mathbb{T}$ -algebra  $A$  with the structure map  $a : \mathbb{T}(A) \rightarrow A$  the element  $a(\mathbb{T}(f)(t)) \in A$ . The category of algebras over a monad  $\mathbb{T}$  is additive/abelian if and only if there are elements (“operations”)  $x + y \in \mathbb{T}(\{x, y\})$ ,  $-x \in \mathbb{T}(\{x\})$ , and  $0 \in \mathbb{T}(\emptyset)$  in  $\mathbb{T}$  satisfying the obvious equations of compatibility with each other and the equations of commutativity with all the other operations in  $\mathbb{T}$  [44, §10]. We will call such a monad  $\mathbb{T} : \mathbf{Set} \rightarrow \mathbf{Set}$  *additive*.

Furthermore, let  $\kappa$  be a regular cardinal. The object  $P \in \mathcal{K}$  is called *abstractly  $\kappa$ -small* if any morphism  $P \rightarrow P^{(S)}$  factors through the coproduct of copies of  $P$  indexed by a subset in  $S$  of cardinality smaller than  $\kappa$  (cf. [43]). In this case the functor  $\mathbb{T} : \mathbf{Set} \rightarrow \mathbf{Set}$  preserves  $\kappa$ -filtered colimits, the object  $P$  is  $\kappa$ -presentable, and the category  $\mathcal{K}$  is locally  $\kappa$ -presentable [1]. Conversely, if the category  $\mathcal{K}$  is locally presentable, then the object  $P$  is  $\kappa$ -presentable for some  $\kappa$ , and hence abstractly  $\kappa$ -small. Monads preserving filtered colimits on the category of sets correspond to finitary algebraic theories [8, Section 4.6]; the categories of algebras over such monads are called the categories of models of algebraic theories in [24, 3]. These are what are called the algebraic monads or “generalized rings” in [15]. The categories of algebras over monads on  $\mathbf{Set}$  preserving  $\kappa$ -filtered colimits were studied in [44]; we will call them the *categories of models of  $\kappa$ -ary algebraic theories*.

1.2. Suppose now that the natural map from the coproduct of any family of projective objects in  $\mathcal{K}$  to their product is a monomorphism; equivalently, it suffices that the natural map  $P^{(S)} \rightarrow P^S$  from the coproduct of any number of copies of the object  $P$  to their product be a monomorphism. Then the set  $\mathbb{T}(S) = \mathcal{K}(P, P^{(S)})$  of all morphisms  $P \rightarrow P^{(S)}$  in  $\mathcal{K}$  maps injectively into the set  $\mathcal{K}(P, P^S)$ . The latter set is bijective to the  $S$ -indexed product of copies of the set of all morphisms  $P \rightarrow P$  in  $\mathcal{K}$ . Denote the associative ring opposite to the ring  $\mathcal{K}(P, P)$  by  $R$ .

So, for any set  $S$ , the set  $\mathbb{T}(S)$  is a subset in  $R^S$ . The  $S$ -ary operation in (the underlying set of) a  $\mathbb{T}$ -algebra  $A$  corresponding to an element of  $\mathbb{T}(S)$  can be now interpreted as an operation of infinite summation of  $S$ -indexed families of elements of  $A$  with the coefficients belonging to  $R$ . The  $S$ -indexed families of coefficients in  $R$  for which such an infinite summation operation exists in  $\mathbb{T}$  (i. e., the  $S$ -indexed families of elements in  $R$  belonging to the subset  $\mathbb{T}(S) \subseteq R^S$ ) we will call  $\mathbb{T}$ -*admissible*. Since we assume  $\mathcal{K}$  to be an abelian category ( $\mathbb{T}$  to be an additive monad), we have  $\mathbb{T}(S) = R^S$  for any finite set  $S$ , that is any finite family of elements in  $R$  is  $\mathbb{T}$ -admissible.

The datum of an abelian category  $\mathcal{K}$  with a projective generator  $P$  such that arbitrary coproducts of copies of  $P$  exist in  $\mathcal{K}$  and map monomorphically into the respective products (or an additive monad  $\mathbb{T} : \mathbf{Set} \rightarrow \mathbf{Set}$  such that  $\mathbb{T}(S) \subseteq \prod_{s \in S} \mathbb{T}(\{s\})$  for all sets  $S$ ) is equivalent to the following set of data:

- a associative ring  $R$  with unit;
- for any set  $S$ , a subset (in fact, an  $R$ - $R$ -subbimodule) of “ $\mathbb{T}$ -admissible families of coefficients”  $\mathbb{T}(S) \subseteq R^S$ ; and
- the map (in fact, an  $R$ - $R$ -bimodule morphism) of “sum of the coefficients in an admissible family”  $\sum_S : \mathbb{T}(S) \rightarrow R$ ,

which has to satisfy the following conditions:

- (i) all the finite families of coefficients are admissible, i. e.,  $\mathbb{T}(S) = R^S$  for a finite set  $S$ ; the map  $\sum_S$  is the conventional finite sum in the ring  $R$  in this case;
- (ii) any subfamily of an admissible family of coefficients is admissible, i. e., for any subset  $S' \subseteq S$ , the projection map  $R^S \rightarrow R^{S'}$  takes  $\mathbb{T}(S)$  into  $\mathbb{T}(S')$ ;
- (iii) given a map of sets  $U \rightarrow S$ , for any family of coefficients  $t \in \mathbb{T}(U)$  the family of sums  $S \ni s \mapsto \sum_{u \in f^{-1}(s)} t(u)$  belongs to  $\mathbb{T}(S)$ , and the iterated sum is equal to the sum of the original family,  $\sum_{s \in S} \sum_{u \in f^{-1}(s)} t(u) = \sum_{u \in U} t(u)$ ;
- (iv) for any family of coefficients  $r \in \mathbb{T}(S)$  and any family of families of coefficients  $t_s \in \mathbb{T}(f^{-1}(s))$  defined for all  $s \in S$ , the family  $U \ni u \mapsto r(f(u))t_{f(u)}(u)$  belongs to  $\mathbb{T}(U)$ , and  $\sum_{s \in S} r(s) \sum_{u \in f^{-1}(s)} t_s(u) = \sum_{u \in U} r(f(u))t_{f(u)}(u)$ .

Given a monad  $\mathbb{T} : \mathbf{Set} \rightarrow \mathbf{Set}$ , the maps  $\sum_S : \mathbb{T}(S) \rightarrow R = \mathbb{T}(\{*\})$  are obtained by applying the functor  $\mathbb{T}$  to the projection maps  $S \rightarrow \{*\}$ . Given a ring  $R$  with the subsets  $\mathbb{T}(S) \subseteq R^S$  and the maps  $\sum_S : \mathbb{T}(S) \rightarrow R$ , the functoriality maps  $\mathbb{T}(f) : \mathbb{T}(U) \rightarrow \mathbb{T}(S)$  are defined in terms of the maps  $\sum_S$ , and the monad multiplication maps  $\mathbb{T}(\mathbb{T}(U)) \rightarrow \mathbb{T}(U)$  are defined in terms of the multiplication operation in the ring  $R$  together with the maps  $\sum_S$ , using the above axioms; while the monad unit maps  $U \rightarrow \mathbb{T}(U)$  are defined in terms of the unit element of the ring  $R$ .

In particular, it follows from the conditions (i–iii) that no nonzero element of  $R$  can occur in an admissible family of coefficients more than a finite number of times. Hence, given a family of coefficients  $t \in \mathbb{T}(S)$ , the cardinality of the set of all  $s \in S$  for which  $t(s) \neq 0$  cannot exceed the cardinality of  $R$ . Thus any abelian category  $\mathcal{K}$  with a projective generator  $P$  such that the natural maps  $P^{(S)} \rightarrow P^S$  are monomorphisms for all sets  $S$  is locally  $\kappa$ -presentable, where  $\kappa$  is the successor cardinal of the cardinality of the set  $R = \mathcal{K}(P, P)$ .

A natural class of examples of monads  $\mathbb{T}$  or abelian categories  $\mathcal{K}$  as above is related to topological rings. Let  $\mathfrak{R}$  be a complete, separated topological ring where open right ideals form a base of neighborhoods of zero. For any set  $S$ , set  $\mathbb{T}(S) = \mathfrak{R}[[S]]$  to be the set of all formal linear combinations of elements of  $S$  with the coefficients in  $\mathfrak{R}$  forming a family converging to zero in the topology of  $\mathfrak{R}$ ; in other words, a family of coefficients  $t \in \mathfrak{R}^S$  belongs to  $\mathbb{T}(S)$  if and only if for any neighborhood

of zero  $\mathfrak{U} \subseteq \mathfrak{R}$  the set of all  $s \in S$  for which  $t(s) \notin \mathfrak{U}$  is finite. The summation map  $\sum_S : \mathfrak{R}[[S]] \rightarrow \mathfrak{R}$  is defined as the infinite sum of the coefficients taken with respect to the topology of  $\mathfrak{R}$ , i. e., the limit of finite partial sums in  $\mathfrak{R}$ . The abelian category of algebras/modules over the monad  $S \mapsto \mathfrak{R}[[S]]$  is called the category of *left contramodules* over topological ring  $\mathfrak{R}$ .

Contramodules were introduced originally by Eilenberg and Moore [16] in the case of coalgebras over commutative rings as module-like objects dual-analogous to, but different from, the more familiar comodules. They were subsequently studied by Barr [4]. The definition of a contramodule was extended to topological rings in [27, Appendices A and D] and [28] (see [30] for an overview).

We do not know whether all additive monads  $\mathbb{T} : \mathbf{Set} \rightarrow \mathbf{Set}$  such that  $\mathbb{T}(S) \subseteq \prod_{s \in S} \mathbb{T}(\{s\})$  for all sets  $S$  come from topological rings. One can try to define a topology on the ring  $R = \mathbb{T}(\{*\})$  by calling a subset  $V \subseteq R$  a neighborhood of zero whenever for any  $t \in \mathbb{T}(S)$  one has  $t(s) \in V$  for all but a finite number of elements  $s \in S$ ; but it is not clear, e. g., why does this topology need to be separated.

Nevertheless, the categories of contramodules over topological rings provide a large class of locally presentable abelian categories with enough projective objects, which play a role in homological algebra, commutative algebra, and representation theory. Apparently, it was precisely the (perceived) lack of relevant examples that discouraged the study of locally presentable abelian categories in general, and abelian categories of models of  $\kappa$ -ary algebraic theories in particular, for many years after the monographs [44] and [1] appeared in print. The present work, motivated originally by the first-named author's interest in the possibilities of applying contemporary set-theoretic techniques of homological algebra and category theory to the categories of contramodules, aims to begin filling this gap.

1.3. Having discussed the place of contramodule categories in the general category theory, let us now say a few words about covers, envelopes, and cotorsion theories. The classical *flat cover conjecture* has two approaches to its solution developed in the literature. In fact, the paper [6], where this conjecture was first proved, contains two proofs. The proof of Bican and El Bashir, based on the approach suggested by Bican and Torrecillas [7], was subsequently generalized by El Bashir in [12] to the claim that, for any accessible full subcategory  $\mathcal{A}$  closed under coproducts and directed colimits in a Grothendieck abelian category  $\mathcal{K}$ , any object of  $\mathcal{K}$  has an  $\mathcal{A}$ -cover. In this paper, we generalize this result even further by replacing a Grothendieck category with a locally presentable (or “ $\kappa$ -Grothendieck”) abelian category.

The second proof of the flat cover conjecture, due to Enochs, was based on the result of Eklof and Trlifaj [17] claiming that any cotorsion theory generated by a set of objects in the category of modules over a ring (or, slightly more generally, in a Grothendieck abelian category with enough projective objects) is complete. As an accessible full subcategory  $\mathcal{A}$  closed under coproducts and directed colimits does not have to be closed under extensions in  $\mathcal{K}$ , while a deconstructible class of objects

containing the projectives and closed under direct summands (the left part  $\mathcal{F}$  of a cotorsion theory  $(\mathcal{F}, \mathcal{C})$  generated by a set  $\mathcal{S}$ ) does not have to be closed under directed colimits, the approaches of Bican–El Bashir and Eklof–Trlifaj complement each other nicely, neither of the two results being a particular case of the other one.

In fact, it is easy to observe that the Eklof–Trlifaj theorem does not require projective objects and can be stated for arbitrary Grothendieck abelian categories  $\mathcal{K}$  as follows.

**Theorem 1.1.** *Let  $\mathcal{S}$  be a set of objects of  $\mathcal{K}$ , and let  $\mathcal{C} = \mathcal{S}^\perp$  and  $\mathcal{F} = {}^\perp\mathcal{C}$  be the corresponding  $\text{Ext}_{\mathcal{K}}^1$ -orthogonal classes. Let  $\text{filt}(\mathcal{S})$  denote the class of all transfinitely iterated extensions, in the sense of the directed colimit, of objects from  $\mathcal{S}$  in  $\mathcal{K}$ . Suppose that every object of  $\mathcal{K}$  is a quotient object of an object of  $\text{filt}(\mathcal{S})$ . Then  $(\mathcal{F}, \mathcal{C})$  is a complete cotorsion theory in  $\mathcal{K}$ , that is any object  $K$  of  $\mathcal{K}$  can be included into short exact sequences*

$$\begin{aligned} 0 &\longrightarrow K \longrightarrow C \longrightarrow F' \longrightarrow 0 \\ 0 &\longrightarrow C' \longrightarrow F \longrightarrow K \longrightarrow 0 \end{aligned}$$

with  $C, C' \in \mathcal{C}$  and  $F', F \in \mathcal{F}$ . Furthermore, the class  $\mathcal{F}$  consists precisely of all the direct summands of object from  $\text{filt}(\mathcal{S})$ .

The assertion of Theorem 1.1 in the form stated above is *not* true for locally presentable abelian categories. The simplest contramodule categories with nonexact functors of filtered colimits, such as the categories of contramodules over the topological rings of  $p$ -adic integers  $\mathbb{Z}_p$  or of the formal power series  $k[[z]]$  in one variable  $z$  over a field  $k$ , allow one to demonstrate counterexamples by making an obvious “bad” choice of a generating object or set of objects  $\mathcal{S}$ . Nevertheless, our version of the Eklof–Trlifaj theorem applicable to any locally presentable abelian category is surprisingly close to the formulation above, the only essential difference is that one has to explicitly require the object of  $\mathcal{K}$  to be a subobject of an object from  $\mathcal{C}$  (in addition to it being a quotient object of an object from  $\mathcal{F}$  or  $\text{filt}(\mathcal{S})$ ). This condition holds automatically in Grothendieck categories, which always have enough injective objects; but locally presentable abelian categories may contain no injectives.

An alternative approach is to work with generating sets  $\mathcal{S}$  or classes  $\mathcal{F}$  consisting of objects which have been shown to behave well with respect to filtered colimits. This is particularly important for our construction of  $\mathcal{C}$ -envelopes in locally presentable abelian categories, which is based entirely on the assumption of preservation of  $\mathcal{F}$ -monomorphisms by directed colimits, at least in the comma-categories  $K \downarrow \mathcal{K}$ . So we have to engage in a homological study of *flat contramodules* over topological rings in order to prove that the derived functors of filtered colimits in contramodule categories vanish on diagrams of flat contramodules. In the end, our results on covers, envelopes, and cotorsion theories in contramodule categories include

- existence of  $\mathcal{A}$ -covers for any accessible full subcategory  $\mathcal{A}$  closed under coproducts and directed colimits;

- in particular, existence of flat covers;
- completeness of the flat cotorsion theory, that is existence of special flat pre-covers and special cotorsion preenvelopes;
- existence of cotorsion envelopes;
- completeness of any cotorsion theory  $(\mathcal{F}, \mathcal{C})$  generated by a set  $\mathcal{S}$  of flat contramodules;
- completeness of any cotorsion theory  $(\mathcal{F}, \mathcal{C})$  generated by a set of contramodules  $\mathcal{S}$  such that every contramodule is a subcontramodule of a contramodule from  $\mathcal{C}$ .

1.4. We discuss  $\kappa$ -Grothendieck abelian categories and develop our version of the Bican–El Bashir approach to covers in Section 2. Our version of the Eklof–Trlifaj theorem is considered in Section 3, with further details following in Section 4. Counterexamples showing that Theorem 1.1 does not hold in locally presentable abelian categories are also demonstrated in Section 4, and envelopes are discussed in Section 3. We recall the basic definitions and constructions related to contramodules in Section 5. Flat contramodules are studied in Section 6. Cotorsion theories in contramodule categories are constructed in Section 7.

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## 2. $\kappa$ -GROTHENDIECK CATEGORIES, PRECOVERS AND COVERS

Recall that a *set of generators* of a category  $\mathcal{K}$  is a set  $\mathcal{G}$  of objects of  $\mathcal{K}$  such that any object  $K$  of  $\mathcal{K}$  is a quotient of a coproduct of objects from  $\mathcal{G}$ . This is clearly equivalent to the condition that for any two distinct morphisms  $f_1, f_2 : K \rightarrow L$  there is a morphism  $g : G \rightarrow K$  with  $G \in \mathcal{G}$  such that  $f_1 g \neq f_2 g$ .

**Definition 2.1.** Let  $\mathcal{K}$  be a cocomplete abelian category with a set of generators and  $\kappa$  a regular cardinal. We say that  $\mathcal{K}$  is  *$\kappa$ -Grothendieck* if  $\kappa$ -filtered colimits are exact in  $\mathcal{K}$ .

Let  $\mathcal{K}$  be a category and  $\lambda$  a regular cardinal. An object  $K$  in  $\mathcal{K}$  is called  *$\lambda$ -presentable* if the functor  $\mathcal{K}(K, -) : \mathcal{K} \rightarrow \mathbf{Set}$  preserves  $\lambda$ -filtered colimits. A category  $\mathcal{K}$  is called  *$\lambda$ -accessible* if  $\lambda$ -filtered colimits exist in  $\mathcal{K}$  and there is a set  $\mathcal{G}$  of  $\lambda$ -presentable objects such that every object of  $\mathcal{K}$  is a  $\lambda$ -filtered colimit of objects from  $\mathcal{G}$ . A functor  $F : \mathcal{K} \rightarrow \mathcal{L}$  is  *$\lambda$ -accessible* if  $\mathcal{K}$  and  $\mathcal{L}$  are  $\lambda$ -accessible categories and  $F$  preserves  $\lambda$ -filtered colimits [1, 25].

A category or functor is *accessible* if it is  $\lambda$ -accessible for some regular cardinal  $\lambda$ . A full subcategory  $\mathcal{K} \subseteq \mathcal{L}$  is *accessibly embedded* if there exists a cardinal  $\lambda$  such

that  $\mathcal{K}$  is closed under  $\lambda$ -filtered colimits in  $\mathcal{L}$ . A category is *locally  $\lambda$ -presentable* if it is  $\lambda$ -accessible and cocomplete. A category is *locally presentable* if it is locally  $\lambda$ -presentable for some regular cardinal  $\lambda$ .

In a locally  $\kappa$ -presentable category,  $\kappa$ -filtered colimits commute with  $\kappa$ -small limits [1, Proposition 1.59], so locally  $\kappa$ -presentable abelian categories are  $\kappa$ -Grothendieck.

**Theorem 2.2.** *Any  $\kappa$ -Grothendieck category is locally presentable.*

*Proof.* Following [43, Remark 2.2], any cocomplete abelian category  $\mathcal{K}$  with a set of generators such that  $\kappa$ -filtered colimits commute with  $\nu$ -small limits in  $\mathcal{K}$  for some regular cardinals  $\nu \leq \kappa$  is a  $\nu$ -localization of the category of models  $\mathcal{M}$  of a certain  $\kappa$ -ary algebraic theory. (In fact, the category  $\mathcal{M}$  is also abelian.) This means that  $\mathcal{K}$  is a reflective full subcategory of  $\mathcal{M}$  such that the left adjoint functor to the inclusion of  $\mathcal{K}$  to  $\mathcal{M}$  preserves  $\nu$ -small limits. In particular,  $\mathcal{M}$  is locally  $\kappa$ -presentable,  $\mathcal{K}$  is a localization of  $\mathcal{M}$  and, following [9, Proposition 6.7],  $\mathcal{K}$  is closed under  $\alpha$ -filtered colimits in  $\mathcal{M}$  for some regular cardinal  $\alpha \geq \kappa$ . Thus  $\mathcal{K}$  is locally  $\alpha$ -presentable (see [1, Remark 1.20 and Theorem 1.39]).  $\square$

**Lemma 2.3.** *Let  $\mathcal{K}$  be a locally  $\lambda$ -presentable category,  $\gamma$  be the number of morphisms between (non-isomorphic)  $\lambda$ -presentable objects of  $\mathcal{K}$  and  $\kappa \geq \lambda$ ,  $\kappa > \gamma$  be a regular cardinal. Then a  $\kappa$ -presentable object of  $\mathcal{K}$  cannot have an increasing chain of subobjects of length  $\kappa$ .*

*Proof.* Let  $\mathcal{K}$  be a locally  $\lambda$ -presentable category and  $\mathcal{X}$  be its representative full subcategory of  $\lambda$ -presentable objects. Consider the canonical full embedding  $E : \mathcal{K} \rightarrow \mathbf{Set}^{\mathcal{X}^{\text{op}}}$  sending  $K$  to  $\mathcal{K}(-, K)$  restricted on  $\mathcal{X}$ . This functor sends any increasing chain of subobjects of  $K$  to the increasing chain of subobjects of  $EK$ . Since any  $\kappa$ -presentable objects of  $\mathcal{K}$  is a  $\kappa$ -small  $\lambda$ -filtered colimit of  $\lambda$ -presentable objects for  $\kappa \geq \lambda$  (see [25, Proposition 2.3.11]), and  $E$  preserves  $\lambda$ -filtered colimits and sends  $\lambda$ -presentable objects to finitely presentable ones,  $E$  sends  $\kappa$ -presentable objects to  $\kappa$ -presentable ones. The forgetful functor  $\mathbf{Set}^{\mathcal{X}^{\text{op}}} \rightarrow \mathbf{Set}^X$ , where  $X$  is the set of objects of  $\mathcal{X}$ , preserves  $\kappa$ -presentable objects when  $\kappa > \gamma$ . This implies that a  $\kappa$ -presentable object in  $\mathcal{K}$  cannot have an increasing chain of subobjects of length  $\kappa$ . In fact, such a chain would yield a similar chain of subobjects of  $EK$ . Applying the forgetful functor, we would get an increasing chain of subsets of the disjoint union of  $EK(A)$  over the objects  $A$  of  $\mathcal{X}$ . Since  $EK$  is  $\kappa$ -presentable, all the sets  $EK(A)$  have cardinality  $< \kappa$  and thus this chain cannot have length  $\kappa$ .  $\square$

Recall that a morphism  $f : K \rightarrow L$  is a  *$\lambda$ -pure epimorphism* if any morphism  $A \rightarrow L$  with  $A$   $\lambda$ -presentable factorizes through  $f$ . In a locally  $\lambda$ -presentable category any  $\lambda$ -pure epimorphism is an epimorphism.

**Definition 2.4.** We say that a category  $\mathcal{K}$  *has enough  $\lambda$ -pure quotients* if for each object  $K$  there is, up to isomorphism, only a set of morphisms  $f : L \rightarrow K$  such that  $f = h \cdot g$ ,  $g$   $\lambda$ -pure epimorphism implies that  $g$  is an isomorphism.

Clearly, if  $\mathcal{K}$  has enough  $\lambda$ -pure quotients then it has enough  $\mu$ -pure quotients for any  $\mu \geq \lambda$ .

In an abelian locally  $\lambda$ -presentable category,  $\lambda$ -pure epimorphisms are precisely cokernels of  $\lambda$ -pure monomorphisms (see [2]). Recall that a monomorphism  $f : K \rightarrow L$  is  $\lambda$ -pure if in every commutative square

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ u \uparrow & & \uparrow v \\ A & \xrightarrow{g} & B \end{array}$$

with  $A$  and  $B$   $\lambda$ -presentable the morphism  $u$  factors through  $g$ .

Thus an abelian locally  $\lambda$ -presentable category has enough  $\lambda$ -pure quotients if and only if for each object  $K$  there is, up to isomorphism, only a set of morphisms  $f : L \rightarrow K$  such that  $fg = 0$  and  $g$   $\lambda$ -pure monomorphism implies that  $g = 0$ . Following [12, Theorem 2.1], any Grothendieck category has enough  $\lambda$ -pure quotients for some regular cardinal  $\lambda$ . As observed in [33], this is valid in any abelian locally presentable category:

**Theorem 2.5.** *Any additive locally presentable category has enough  $\lambda$ -pure quotients for some regular cardinal  $\lambda$ .*  $\square$

An argument deducing Theorem 2.5 from its particular case when the category in question is the category of modules over a ring with several objects (hence a Grothendieck category) can be found in [23, Corollary 1].

Recall that a full subcategory  $\mathcal{A}$  of a category  $\mathcal{K}$  is called *weakly coreflective* if each object  $K$  in  $\mathcal{K}$  has a weak coreflection, i.e., a morphism  $c_K : K^* \rightarrow K$  where  $K^*$  is in  $\mathcal{A}$  such that every morphism  $f : A \rightarrow K$  with  $A$  in  $\mathcal{A}$  factorizes (not necessarily uniquely) through  $c_K$ . Every weakly coreflective subcategory closed under retracts is closed under coproducts in  $\mathcal{K}$ . The following result was proved in [33, Proposition 2.3].

**Proposition 2.6.** *Let  $\mathcal{K}$  be a locally presentable category having enough  $\lambda$ -pure quotients for some regular cardinal  $\lambda$ . Let  $\mathcal{A}$  be an accessible full subcategory of  $\mathcal{K}$  which is closed under coproducts and directed colimits. Then  $\mathcal{A}$  is weakly coreflective in  $\mathcal{K}$ .*  $\square$

In the module-theoretic terminology,  $\mathcal{A}$  being weakly coreflective means that any object of  $\mathcal{K}$  has an  $\mathcal{A}$ -precover. If, in addition, any morphism  $h : K^* \rightarrow K^*$  with  $c_K h = c_K$  is an isomorphism, we say that  $c_K : K^* \rightarrow K$  is an  $\mathcal{A}$ -cover. The category-theoretic terminology is that  $\mathcal{A}$  is *stably weakly coreflective* (see [33]).

The following theorem is a state-of-the-art version of [18, Theorem 3.1].

**Theorem 2.7.** *Let  $\mathcal{K}$  be a locally presentable category and  $\mathcal{A}$  its weakly coreflective full subcategory which is closed under directed colimits. Then  $\mathcal{A}$  is stably weakly coreflective.*



*Proof.* Let  $\mathcal{K}$  be a locally  $\lambda$ -presentable category. Since  $\lambda$ -filtered colimits commute with finite limits, the squaring functor  $-^2 : \mathcal{K} \rightarrow \mathcal{K}$  (sending  $K$  to  $K \times K$ ) preserves  $\lambda$ -filtered colimits. Thus it preserves  $\mu$ -presentable objects for some  $\mu \geq \lambda$  (see [1, 2.19]). Let  $K$  be in  $\mathcal{K}$  and assume that  $K$  does not have a stable weak coreflection to  $\mathcal{A}$ . Let  $c_K : K^* \rightarrow K$  be a weak coreflection. There is a regular cardinal  $\kappa$  such that  $\kappa \triangleright \lambda$ ,  $\kappa \geq \mu$  and  $\kappa > \gamma$  for which  $K^*$  is  $\kappa$ -presentable (see [1, 2.13(6)]). Then the squaring functor preserves  $\kappa$ -presentable objects, and a  $\kappa$ -presentable object cannot have an increasing chain of subobjects of length  $\kappa$  (see Lemma 2.3).

We will form the smooth chain  $(t_{ij} : T_i \rightarrow T_j)_{i < j \leq \kappa}$  and a compatible cocone  $c_i : T_i \rightarrow K$  starting with  $T_0 = K^*$  and  $c_0 = c_K$ . Having  $c_i : T_i \rightarrow K$ , we put  $T_{i+1} = T_i$  and  $c_{i+1} = c_i$ . If there is  $f : T_i \rightarrow T_i$  with  $c_i f = c_i$  which is not a monomorphism, we put  $t_{i,i+1} = f$ . If all morphisms  $f : T_i \rightarrow T_i$  with  $c_i f = c_i$  are monomorphisms, we take  $t_{i,i+1}$  with  $c_i t_{i,i+1} = c_i$  which is not an isomorphism. Since  $c_i : T_i \rightarrow K$  is not a stable weak coreflection, such  $t_{i,i+1}$  exists. All objects  $T_i$ ,  $i < \kappa$  are  $\kappa$ -presentable.

Let us show that the ordinals  $i < \kappa$  such that  $t_{i\kappa}$  is a monomorphism are cofinal in  $\kappa$ . Consider kernel pairs  $s_j, r_j : S_j \rightarrow T_j \times T_j$  of  $t_{j\kappa}$  and kernel pairs  $s_{ji}, r_{ji} : S_{ji} \rightarrow T_j \times T_j$  of  $t_{ji}$ ,  $j < i$ . Since  $\kappa$ -directed colimits commute with finite limits, we get that  $S_j$  is the  $\kappa$ -directed colimit of  $S_{ji}$ ,  $j < i < \kappa$ . Since  $T_j \times T_j$  does not contain an increasing chain of subobjects of length  $\kappa$ , there is  $j < i < \kappa$  such that  $S_j = S_{ji}$ . We form a smooth sequence  $i_0 < i_1 < \dots < i_k < \dots$ ,  $k \leq \lambda$  by putting  $i_0 = j$  and  $S_{i_{k+1}} = S_{i_k, i_{k+1}}$ . Since  $\lambda$ -directed colimits commute with  $\lambda$ -small limits, we have

$$S_{i_\lambda} = \operatorname{colim}_{k < \lambda} S_{i_k} = \operatorname{colim}_{k < \lambda} S_{i_k, i_{k+1}} = T_{i_\lambda}.$$

Hence  $t_{i_\lambda, \kappa}$  is a monomorphism.

If  $t_{i\kappa}$  is a monomorphism then  $t_{ij}$  is a monomorphism for all  $i < j \leq \kappa$ . In particular,  $t_{i,i+1}$  is a monomorphism and, following the construction, it is not an isomorphism. Moreover, if  $t_{ij}$  is an isomorphism for some  $i < j \leq \kappa$ , then  $t_{i+1,j}$  is not a monomorphism and  $t_{ij}^{-1} t_{i+1,j}$  is not a monomorphism, which contradicts the condition that all morphisms  $f : T_i \rightarrow T_i$  with  $c_i f = c_i$  are monomorphisms (we recall that  $T_{i+1} = T_i$  and  $c_{i+1} = c_i$ ). Thus  $t_{ij}$  cannot be an isomorphism. We have shown that  $T_\kappa$  contains an increasing chain of subobjects of length  $\kappa$ .

Since  $c_0 : T_0 \rightarrow K$  is a weak coreflection, there is a morphism  $h : T_\kappa \rightarrow T_0$  such that  $c_0 h = c_\kappa$ . Once again, if  $t_{i\kappa}$  is a monomorphism then, following the construction, all morphisms  $f : T_i \rightarrow T_i$  such that  $c_i f = c_i$  are monomorphisms. Hence  $h t_{i\kappa}$  is a monomorphism because we have a monomorphism

$$T_i \xrightarrow{h t_{i\kappa}} T_0 \xrightarrow{t_{0i}} T_i.$$

Consequently,  $h$  is a monomorphism and we get an increasing chain of subobjects of  $T_0$  of length  $\kappa$ , which is a contradiction.  $\square$

**Corollary 2.8.** *Let  $\mathcal{K}$  be an locally presentable abelian category and  $\mathcal{A}$  its accessible full subcategory closed under coproducts and directed colimits. Then  $\mathcal{A}$  is stably weakly coreflective.*  $\square$

*Proof.* Combine Theorem 2.5 and Proposition 2.6 with Theorem 2.7.  $\square$

**Remark 2.9.** Assuming Vopěnka's principle, any full subcategory of a locally presentable category closed under directed colimits is accessible (see [1, Theorem 6.17]). Thus any full subcategory of an locally presentable abelian category closed under coproducts and directed colimits is stably weakly coreflective.

### 3. COTORSION THEORIES, SPECIAL PREENVELOPES AND ENVELOPES

Given two classes of morphisms  $\mathcal{L}$  and  $\mathcal{R}$  in a category  $\mathcal{K}$ , we denote by  $\mathcal{L}^\square$  the class of all morphisms in  $\mathcal{K}$  having the right lifting property with respect to every morphism from  $\mathcal{L}$ , and by  ${}^\square\mathcal{R}$  the class of all morphisms having the left lifting property with respect to every morphisms from  $\mathcal{R}$ . We also denote by  $\mathcal{L}^\Delta$  the class of all objects in  $\mathcal{K}$  that are injective to every morphism from  $\mathcal{L}$ , and by  ${}^\Delta\mathcal{R}$  the class of all objects that are projective to every morphism in  $\mathcal{R}$  (see [32, Definition 2.1]).

A pair of classes of morphisms  $(\mathcal{L}, \mathcal{R})$  in a category  $\mathcal{K}$  is called a *weak factorization system* if  $\mathcal{R} = \mathcal{L}^\square$ ,  $\mathcal{L} = {}^\square\mathcal{R}$ , and every morphism  $f$  in  $\mathcal{K}$  can be decomposed as  $f = rl$  with  $l \in \mathcal{L}$  and  $r \in \mathcal{R}$  [32, Definition 2.2]. The notion of a weak factorization system can be viewed as “a half of” a model structure: in a (closed) model category, the pairs of classes of morphisms (cofibrations, trivial fibrations) and (trivial cofibrations, fibrations) are weak factorization systems.

Let  $\mathcal{K}$  be an abelian category and  $\mathcal{A}, \mathcal{B}$  its full subcategories. Denote by  $\mathcal{A}^\perp$  the class of all objects  $B \in \mathcal{K}$  such that  $\text{Ext}_{\mathcal{K}}^1(A, B) = 0$  for all  $A \in \mathcal{A}$ . Similarly, the class  ${}^\perp\mathcal{B}$  consists of all objects  $A \in \mathcal{K}$  such that  $\text{Ext}_{\mathcal{K}}^1(A, B) = 0$  for all  $B \in \mathcal{B}$ . The class  $\mathcal{A}^\perp$  is always closed under products in  $\mathcal{K}$  and the class  ${}^\perp\mathcal{B}$  is closed under coproducts.

An  *$\mathcal{A}$ -monomorphism* is defined as a monomorphism in  $\mathcal{K}$  whose cokernel belongs to  $\mathcal{A}$ . The class of all  $\mathcal{A}$ -monomorphisms will be denoted by  $\mathcal{A}\text{-Mono}$ . Dually, the class of all epimorphisms whose kernel is in  $\mathcal{B}$  is denoted as  $\mathcal{B}\text{-Epi}$  and these epimorphisms are called  *$\mathcal{B}$ -epimorphisms*. Like in [32, Remark 4.3], an object  $B$  is injective to all  $\mathcal{A}$ -monomorphisms if and only if  $\text{Ext}_{\mathcal{K}}^1(A, B) = 0$  for all  $A \in \mathcal{A}$ . Thus  $\mathcal{A}\text{-Mono}^\Delta = \mathcal{A}^\perp$  and  ${}^\Delta\mathcal{B}\text{-Epi} = {}^\perp\mathcal{B}$ .

**Lemma 3.1.** *Let  $\mathcal{K}$  be an abelian category and  $\mathcal{A}$  a full subcategory of  $\mathcal{K}$  such that any object of  $\mathcal{K}$  is a quotient of an object from  $\mathcal{A}$ . Then  $g$  belongs to  $\mathcal{A}\text{-Mono}^\square$  if and only if  $g$  is an epimorphism whose kernel is in  $\mathcal{A}^\perp$ .*

*Proof.* Let  $g : C \rightarrow D$  belong to  $\mathcal{A}\text{-Mono}^\square$ . Following the proof of [32, Lemma 4.4], the kernel of  $g$  belongs to  $\mathcal{A}^\perp$ . There is  $A \in \mathcal{A}$  and an epimorphism  $v : A \rightarrow D$ .

Consider the square

$$\begin{array}{ccc} 0 & \longrightarrow & C \\ \downarrow & & \downarrow g \\ A & \xrightarrow{v} & D \end{array}$$

Since  $0 \rightarrow A$  is an  $\mathcal{A}$ -monomorphism, there is  $t : A \rightarrow C$  such that  $gt = v$ . Thus  $g$  is an epimorphism.

The converse is [27, Lemma 1 in Section 9.1].  $\square$

Given a class of morphisms  $\mathcal{L}$  in a cocomplete category  $\mathcal{K}$ , let  $\text{cof}(\mathcal{L})$  denote the closure of  $\mathcal{L}$  with respect to pushouts, transfinite compositions, and retracts. For any class of morphisms  $\mathcal{R}$  in  $\mathcal{K}$ , one has  $\text{cof}(\square\mathcal{R}) = \square\mathcal{R}$ .

**Lemma 3.2.** *Let  $\mathcal{B}$  be a full subcategory of a cocomplete abelian category  $\mathcal{K}$  such that any object of  $\mathcal{K}$  is a subobject of an object from  $\mathcal{B}$ . Then  $\text{cof}({}^\perp\mathcal{B}\text{-Mono}) = {}^\perp\mathcal{B}\text{-Mono}$ .*

*Proof.* Following the dual of Lemma 3.1,  ${}^\perp\mathcal{B}\text{-Mono} = \square(\mathcal{B}\text{-Epi})$ .  $\square$

Recall that a pair of classes of morphisms  $(\mathcal{F}, \mathcal{C})$  in an abelian category  $\mathcal{K}$  is called a *cotorsion theory* if  $\mathcal{C} = \mathcal{F}^\perp$  and  $\mathcal{F} = {}^\perp\mathcal{C}$ . A cotorsion theory is called *complete* if any object  $K$  has a *special  $\mathcal{F}$ -precover*, that is a  $\mathcal{C}$ -epimorphism  $F \rightarrow K$  with  $F \in \mathcal{F}$ , and a *special  $\mathcal{C}$ -preenvelope*, i. e., an  $\mathcal{F}$ -monomorphism  $K \rightarrow C$  with  $C \in \mathcal{C}$ .

Since the objects of  $\mathcal{F}$  are projective with respect to  $\mathcal{C}$ -epimorphisms, any special  $\mathcal{F}$ -precover of an object  $K$  is an  $\mathcal{F}$ -precover, that is a weak coreflection of  $K$  to  $\mathcal{F}$ . Dually, any special  $\mathcal{C}$ -preenvelope is a  $\mathcal{C}$ -preenvelope, i. e., a weak reflection to  $\mathcal{C}$  (see also [45, Propositions 2.1.3-4]).

**Lemma 3.3.** *Let  $(\mathcal{F}, \mathcal{C})$  be a cotorsion theory in an abelian category  $\mathcal{K}$  such that any object of  $\mathcal{K}$  is a quotient of an object from  $\mathcal{F}$  and a subobject of an object from  $\mathcal{C}$ . Then  $\mathcal{F}\text{-Mono}^\square = \mathcal{C}\text{-Epi}$  and  $\square\mathcal{C}\text{-Epi} = \mathcal{F}\text{-Mono}$ .*

*Proof.* It follows from Lemma 3.1 and its dual.  $\square$

A cotorsion theory  $(\mathcal{F}, \mathcal{C})$  is *generated* by a set  $\mathcal{S}$  if  $\mathcal{C} = \mathcal{S}^\perp$ .

**Lemma 3.4.** *Let  $\mathcal{K}$  be a locally  $\lambda$ -presentable abelian category and  $\mathcal{S}$  be a set of  $\lambda$ -presentable objects in  $\mathcal{K}$ . Then any  $\mathcal{S}$ -monomorphism in  $\mathcal{K}$  is a pushout of an  $\mathcal{S}$ -monomorphism with a  $\lambda$ -presentable codomain.*

*Proof.* Let  $K \rightarrow L$  be a monomorphism with a cokernel  $S \in \mathcal{S}$ . Let the object  $L$  be the colimit of a  $\lambda$ -filtered diagram of  $\lambda$ -presentable objects  $L_i$ . Denote by  $K_i$  the kernels of the compositions  $L_i \rightarrow L \rightarrow S$ . Since  $\lambda$ -filtered colimits commute with kernels in  $\mathcal{K}$ , we have  $K = \text{colim } K_i$ . Hence  $S = \text{colim}(L_i/K_i)$ . Since the object  $S$  is  $\lambda$ -presentable, we can conclude that there exists  $i$  for which the morphism  $L_i/K_i \rightarrow S$

is a split epimorphism. Thus the morphism  $L_i \rightarrow S$  is an epimorphism. We have constructed a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & S \longrightarrow 0 \\ & & \uparrow & & \uparrow & \nearrow & \\ 0 & \longrightarrow & K_i & \longrightarrow & L_i & & \end{array}$$

where the two horizontal sequences are exact. It follows that the morphism  $K \rightarrow L$  is a pushout of the morphism  $K_i \rightarrow L_i$ .

Indeed, let  $K \rightarrow L'$  be the pushout of  $K_i \rightarrow L_i$  by  $K_i \rightarrow K$ ; then there is the induced morphism  $L' \rightarrow L$  and a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & S \longrightarrow 0 \\ & & \searrow & & \uparrow & \nearrow & \\ & & & & L' & & \end{array}$$

of a morphism between two short exact sequences with the same subobject and the same quotient object. A simple diagram chase now shows that both the kernel and the cokernel of the morphism  $L \rightarrow L'$  vanish. Since the category  $\mathcal{K}$  is abelian, we can conclude that  $L \rightarrow L'$  is an isomorphism.  $\square$

**Proposition 3.5.** *Let  $\mathcal{K}$  be a locally presentable abelian category and  $(\mathcal{F}, \mathcal{C})$  a cotorsion theory generated by a set. Suppose that any object of  $\mathcal{K}$  is a quotient of an object from  $\mathcal{F}$  and a subobject of an object from  $\mathcal{C}$ . Then  $(\mathcal{F}\text{-Mono}, \mathcal{C}\text{-Epi})$  is a weak factorization system.*

*Proof.* We can assume that  $\mathcal{K}$  is locally  $\lambda$ -presentable and any object from a generating set  $\mathcal{S}_0$  of  $(\mathcal{F}, \mathcal{C})$  is  $\lambda$ -presentable. Any  $\lambda$ -presentable object  $X$  is a quotient of an object  $A_X$  from  $\mathcal{F}$ . Add to the set  $\mathcal{S}_0$  the objects  $A_X$  for  $X$   $\lambda$ -presentable. We get the set  $\mathcal{S}$  which generates  $(\mathcal{F}, \mathcal{C})$  and has the property that any object of  $\mathcal{K}$  is a quotient object of a coproduct of objects from  $\mathcal{S}$ .

Let the set  $\mathcal{I}$  consist of all  $\mathcal{S}$ -monomorphisms having  $\lambda$ -presentable codomains. We have  $\mathcal{I}^\square \supseteq \mathcal{F}\text{-Mono}^\square = \mathcal{C}\text{-Epi}$ . Any  $\mathcal{S}_0$ -monomorphism in  $\mathcal{K}$  is a pushout of a morphism from  $\mathcal{I}$  (see Lemma 3.4), so  $\mathcal{I}^\square \subseteq \mathcal{S}_0\text{-Mono}^\square$ . Following the proof of Lemma 3.1, the kernel of any  $g : C \rightarrow D$  from  $\mathcal{I}^\square$  belongs to  $\mathcal{C}$ .

Let us show that  $g$  is an epimorphism. There is an epimorphism  $t : \coprod_{i \in I} X_i \rightarrow D$  where  $X_i$ ,  $i \in I$  are  $\lambda$ -presentable. There are epimorphisms  $v_i : A_i \rightarrow X_i$  where  $A_i \in \mathcal{S}$ . Let  $d_i : A_i \rightarrow C$  be diagonals in

$$\begin{array}{ccc} 0 & \longrightarrow & C \\ \downarrow & & \downarrow g \\ A_i & \xrightarrow{t_i v_i} & D \end{array}$$

where  $t_i : X_i \rightarrow D$  is the composition of  $t$  with the coproduct injection. Let  $d : \coprod A_i \rightarrow C$  be induced by  $d_i$ . Then  $gd = tv$  and, since  $v = \coprod_{i \in I} v_i$  is an epimorphism,  $g$  is an epimorphism. Hence  $\mathcal{I}^\square = \mathcal{C}\text{-Epi}$ . Thus  $(\mathcal{F}\text{-Mono}, \mathcal{C}\text{-Epi})$  is a weak factorization system (see [5, Proposition 1.3] or [32, Theorem 2.5]).  $\square$

**Corollary 3.6.** *Let  $\mathcal{K}$  be a locally presentable abelian category and  $(\mathcal{F}, \mathcal{C})$  a cotorsion theory generated by a set. Suppose that any object of  $\mathcal{K}$  is a quotient of an object from  $\mathcal{F}$  and a subobject of an object from  $\mathcal{C}$ . Then  $(\mathcal{F}, \mathcal{C})$  is complete.*

**Corollary 3.7.** *Let  $\mathcal{K}$  be a locally presentable abelian category and  $(\mathcal{F}, \mathcal{C})$  a complete cotorsion theory in  $\mathcal{K}$  such that  $\mathcal{F}$  is closed under directed colimits. Then  $\mathcal{F}$  is stably weakly coreflective in  $\mathcal{K}$ .*

*Proof.* This is a particular case of Theorem 2.7.  $\square$

**Remark 3.8.** Since any object has a special  $\mathcal{F}$ -precover and an  $\mathcal{F}$ -cover is a retract of any  $\mathcal{F}$ -precover, an  $\mathcal{F}$ -cover from Corollary 3.7 is special (see also the proof of Wakamatsu's lemma [45, Lemma 2.1.1]).

**Lemma 3.9.** *Let  $(\mathcal{L}, \mathcal{R})$  be a weak factorization system in a category  $\mathcal{K}$  and  $K$  an object in  $\mathcal{K}$ . Let  $\mathcal{M}$  be the full subcategory of the comma-category  $K \downarrow \mathcal{K}$  consisting of the morphisms  $K \rightarrow X$  belonging to  $\mathcal{L}$ . Then  $\mathcal{M}$  is weakly coreflective in  $K \downarrow \mathcal{K}$ .*

*Proof.* Decompose a morphism  $f : K \rightarrow X$  as  $f = rl$ , where  $l : K \rightarrow Z$  belongs to  $\mathcal{L}$  and  $r : Z \rightarrow X$  belongs to  $\mathcal{R}$ . Let  $m : K \rightarrow Y$  be an object of  $\mathcal{M}$  and  $t : Y \rightarrow X$  be a morphism from  $m$  to  $f$  in  $K \downarrow \mathcal{K}$ . So we have a commutative square

$$\begin{array}{ccc} K & \xrightarrow{l} & Z \\ m \downarrow & & \downarrow r \\ Y & \xrightarrow{t} & X \end{array}$$

According to the lifting property of  $m \in \mathcal{L}$  with respect to  $r \in \mathcal{R}$ , there exists a morphism  $h : Y \rightarrow Z$  in  $\mathcal{K}$  such that  $l = gm$  and  $t = rh$ . We have proved that  $r : l \rightarrow f$  is a weak coreflection of  $f$  to  $\mathcal{M}$  in  $K \downarrow \mathcal{K}$ .  $\square$

We say that  $\mathcal{F}$  is *strongly closed under directed colimits* in  $\mathcal{K}$  if  $\mathcal{F}$ -monomorphisms are closed under directed colimits in any comma-category  $K \downarrow \mathcal{K}$ . It implies that  $\mathcal{F}$  is closed under directed colimits. In a Grothendieck category, the two concepts are equivalent.

The following corollary is a version of [45, Theorems 2.2.6 and 3.4.3–4] that works in locally presentable abelian categories.

**Corollary 3.10.** *Let  $\mathcal{K}$  be a locally presentable abelian category and  $(\mathcal{F}, \mathcal{C})$  a complete cotorsion theory in  $\mathcal{K}$  such that  $\mathcal{F}$  is strongly closed under directed colimits. Then  $\mathcal{C}$  is stably weakly reflective in  $\mathcal{K}$ .*

*Proof.* Following [33, Proposition 3.5], it suffices to show that for every object  $K$  in  $\mathcal{K}$  the morphism  $K \rightarrow 0$  has a stable weak coreflection to the full subcategory  $\mathcal{M}$  of  $K \downarrow \mathcal{K}$  consisting of  $\mathcal{F}$ -monomorphisms. According to Lemma 3.9, any special  $\mathcal{C}$ -preenvelope of  $K$  is a weak coreflection of  $K \rightarrow 0$  to  $\mathcal{M}$ . Following Theorem 2.7,  $K \rightarrow 0$  has a stable weak coreflection to  $\mathcal{M}$ .  $\square$

**Remark 3.11.** Since any object has a special  $\mathcal{C}$ -preenvelope and a  $\mathcal{C}$ -envelope is a retract of any  $\mathcal{C}$ -preenvelope, a  $\mathcal{C}$ -envelope from Corollary 3.10 is special (see also the proof of [45, Lemma 2.1.2]).

**Remark 3.12.** Following [20, Proposition 5.4],  $(\mathcal{F}\text{-Mono}, \mathcal{C}\text{-Epi})$  is a weak factorization system for any complete cotorsion theory  $(\mathcal{F}, \mathcal{C})$  in an abelian category  $\mathcal{K}$ . So, for any object  $K$  in  $\mathcal{K}$  the subcategory  $\mathcal{M}$  is weakly coreflective in  $K \downarrow \mathcal{K}$  by Lemma 3.9. When  $\mathcal{K}$  is locally presentable and  $\mathcal{F}$  is strongly closed under directed colimits, Theorem 2.7 allows to conclude that  $\mathcal{M}$  is stably weakly coreflective. Applying [33, Proposition 3.5], one proves, in the assumptions of Corollary 3.10, that  $(\mathcal{F}\text{-Mono}, \mathcal{C}\text{-Epi})$  is a stable weak factorization system in  $\mathcal{K}$ .

#### 4. FURTHER DETAILS ON COTORSION THEORIES

The following counterexamples show that Theorem 1.1 does not hold for locally presentable/ $\kappa$ -Grothendieck abelian categories in general (see Corollary 3.6, Theorem 4.8, or Proposition 4.16 for formulations applicable to such categories).

**Examples 4.1.** (1) Let  $\mathcal{K}$  be an abelian category and  $\mathcal{T} \subseteq \mathcal{K}$  be a class of objects of projective dimension at most 1, that is  $\text{Ext}_{\mathcal{K}}^n(T, K) = 0$  for all objects  $T \in \mathcal{T}$  and  $K \in \mathcal{K}$  and all  $n \geq 2$ . Denote by  $\mathcal{P} = \mathcal{T}^{\perp_{0,1}} \subseteq \mathcal{K}$  the full subcategory formed by all objects  $P$  in  $\mathcal{K}$  such that  $\text{Ext}_{\mathcal{K}}^n(T, P) = 0$  for all  $T \in \mathcal{T}$  and  $n \geq 0$ .

Clearly,  $\mathcal{P}$  is closed under products and extensions in  $\mathcal{K}$ . Now let  $f : P \rightarrow Q$  be a morphism in  $\mathcal{P}$ ; denote its image by  $Z$ . Then for any  $T \in \mathcal{T}$  one has  $\mathcal{K}(T, Z) = 0$ , since  $Z$  is a subobject of  $Q$ , and  $\text{Ext}_{\mathcal{K}}^1(T, Z) = 0$ , since  $Z$  is a quotient of  $P$ . Thus  $Z$  is in  $\mathcal{P}$ , and it follows that the kernel and cokernel of  $f$  also belong to  $\mathcal{P}$ . We have proven that  $\mathcal{P}$  is an abelian category and the inclusion  $\mathcal{P} \rightarrow \mathcal{K}$  is an exact functor.

(2) Assume that  $\mathcal{K}$  is a locally  $\lambda$ -presentable abelian category and  $\mathcal{T}$  is a set of  $\lambda$ -presentable objects. Let  $\mu$  be a regular cardinal such that all subobjects of  $\lambda$ -presentable objects in  $\mathcal{K}$  are  $\mu$ -presentable. Then one can see from Lemma 3.4 that  $\mathcal{P}$  is closed under  $\mu$ -filtered colimits in  $\mathcal{K}$ . Following [1, Theorem and Corollary 2.48], the category  $\mathcal{P}$  is locally  $\mu$ -presentable and reflective in  $\mathcal{K}$ .

(3) Let  $R$  be a commutative ring,  $\mathcal{K} = R\text{-Mod}$  the category of  $R$ -modules,  $I \subset R$  an ideal, and  $r_i \in I$  a set of generators of  $I$ . Let the set of  $R$ -modules  $\mathcal{T}$  consist of the localizations  $R[r_i^{-1}]$  of the ring  $R$  with respect to its elements  $r_i$ . Then the subcategory  $\mathcal{P} = \mathcal{T}^{\perp_{0,1}} \subset R\text{-Mod}$  is a locally  $\aleph_1$ -presentable abelian category with enough projective objects. In fact, its projective generator  $P$  can be obtained as the reflection of the free  $R$ -module  $R \in \mathcal{K}$  to  $\mathcal{P}$ .

When the ideal  $I$  is generated by a single nonzero-dividing element  $r \in R$ , the functor  $\mathcal{K} \rightarrow \mathcal{P}$  left adjoint to the inclusion can be computed as  $\text{Ext}_R^1(R[r^{-1}]/R, -)$ . When  $R$  is a Noetherian ring, or  $I = (r)$  with  $r$  a nonzero-divisor, the  $R$ -module  $P$  can be computed as the  $I$ -adic completion  $\lim_m R/I^m$  of  $R$ . The full subcategory  $\mathcal{P} \subseteq \mathcal{K}$  does not depend on the choice of a set of generators  $r_i$  of the ideal  $I$ , but only on the ideal itself. It is called the category of  *$I$ -contramodule  $R$ -modules* in [31].

(4) Now let  $I = (r)$  be the principal ideal generated by a noninvertible nonzero-dividing element  $r$  and  $\mathcal{T}$  be the set consisting of a single  $R$ -module  $T = R[r^{-1}]$ . Let the set of objects  $\mathcal{S}$  consist of the projective generator  $P \in \mathcal{P}$  and the  $R$ -module  $S = R/rR$ ; since  $r$  acts by zero in  $S$  and invertibly in  $T$ , one has  $\text{Ext}_R^n(T, S) = 0$  for all  $n \geq 0$ , so  $S \in \mathcal{P}$ . Let  $\mathcal{F}, \mathcal{C} \subseteq \mathcal{P}$  be the corresponding  $\text{Ext}^1$ -orthogonal classes,  $\mathcal{C} = \mathcal{S}^\perp$  and  $\mathcal{F} = {}^\perp\mathcal{C}$ , in the abelian category  $\mathcal{P}$ .

Note that the functors  $\text{Ext}_R^1$  and  $\text{Ext}_{\mathcal{P}}^1$  agree on  $\mathcal{P}$ , as  $\mathcal{P}$  is closed under extensions in  $R\text{-Mod}$ . For any  $R$ -module  $M$ , one has  $\text{Ext}_R^1(S, M) = M/rM$ . So  $\text{Ext}_R^1(S, M) = 0$  means  $M = rM$ , which implies surjectivity of the natural map  $\mathcal{K}(T, M) \rightarrow M$ . For  $M \in \mathcal{P}$ , this is only possible when  $M = 0$ . We have shown that  $\mathcal{C} = 0$  (in particular, there are no nonzero injective objects in  $\mathcal{P}$ ). Hence  $\mathcal{F} = \mathcal{P}$ .

Consequently, nonzero objects of  $\mathcal{P}$  (such as, e. g.,  $S$  or  $P = \lim_m R/r^m R$ ) cannot have special  $\mathcal{C}$ -preenvelopes and the assertion of Theorem 1.1 fails for the locally  $\aleph_1$ -presentable abelian category  $\mathcal{P}$  with the set of objects  $\mathcal{S} = \{S, P\}$ .

The following lemma is a more precise formulation of the assertions contained in (the proof of) Lemma 3.1.

**Lemma 4.2.** *Let  $\mathcal{K}$  be an abelian category and  $\mathcal{A}$  a class of objects in  $\mathcal{K}$ . Then*

- (a) *the class of morphisms  $\mathcal{A}^\perp\text{-Epi}$  is contained in  $\mathcal{A}\text{-Mono}^\square$ ;*
- (b) *the kernel of any morphism from  $\mathcal{A}\text{-Mono}^\square$  belongs to  $\mathcal{A}^\perp$ ;*
- (c) *any morphism  $X \rightarrow Y$  in  $\mathcal{K}$  which has right lifting property with respect to all morphisms  $0 \rightarrow A$ ,  $A \in \mathcal{A}$  and whose codomain  $Y$  is a quotient of an object from  $\mathcal{A}$  is an epimorphism.*  $\square$

**Definition 4.3.** Let  $\alpha$  be an ordinal and  $S_i$  a family of objects in a cocomplete abelian category  $\mathcal{K}$  indexed by the ordinals  $i < \alpha$ . We say that an object  $F$  in  $\mathcal{K}$  is a *transfinitely iterated extension* of the objects  $S_i$  (in the sense of the directed colimit) if there exists a smooth chain  $(f_{ij} : F_i \rightarrow F_j)_{i < j \leq \alpha}$  in  $\mathcal{K}$  such that  $F_0 = 0$ ,  $F_\alpha = F$ , and  $f_{i,i+1}$  is a monomorphism with the cokernel isomorphic to  $S_i$  for every  $i < \alpha$ .

Note that no exactness condition on the directed colimits is imposed in Definition 4.3. If  $F$  is a transfinitely iterated extension of  $S_i$ ,  $i < \alpha$  in  $\mathcal{K}$  and an ordinal  $\beta < \alpha$  is given, then one can claim that  $F_\beta$  is a transfinitely iterated extension of  $S_j$ ,  $j < \beta$  and the cokernel  $F_{\geq \beta}$  of the morphism  $f_{\beta\alpha}$  is a transfinitely iterated extension of  $S_k$ ,  $\beta \leq k < \alpha$ , so there is an exact sequence

$$F_\beta \xrightarrow{f_{\beta\alpha}} F \longrightarrow F_{\geq \beta} \longrightarrow 0.$$

Furthermore,  $F$  is a transfinitely iterated extension of the sequence of objects  $F_\beta$ ,  $S_\beta$ ,  $S_{\beta+1}$ ,  $\dots$ ,  $S_i$ ,  $\dots$ , where  $i < \alpha$ . But the morphism  $f_{\beta\alpha}$  does not need to be a monomorphism, as the following examples illustrate.

**Examples 4.4.** Let  $R$  be a commutative ring and  $r \in R$  be a nonzero-dividing element; set  $S = R/rR$ . Then in the abelian category  $\mathcal{P}$  from Examples 4.1(3–4) the zero object is a transfinitely iterated extension of the sequence of objects  $S$ ,  $S$ ,  $S$ ,  $\dots$  indexed by the nonnegative integers  $n$ . Indeed, set  $F_n = R/r^n R \in \mathcal{P}$ , and let  $f_{ij} : F_i \rightarrow F_j$  be the monomorphism taking  $1 + r^i R$  to  $r^{j-i} + r^j R$ . Clearly,  $\text{coker}(f_{i,i+1}) = S$ . We claim that  $\text{colim}_n F_n = 0$  in  $\mathcal{P}$ .

Indeed, let  $f_n : F_n \rightarrow F$  be a compatible cocone in  $\mathcal{P}$ ; then it can be also viewed as a compatible cocone in  $R\text{-Mod}$ . The colimit of the diagram  $(f_{ij} : F_i \rightarrow F_j)$  in the category of  $R$ -modules is  $R[r^{-1}]/R$ , so the morphisms  $f_n$  factor through an  $R$ -module morphism  $f : R[r^{-1}]/R \rightarrow F$ . Now any morphism  $R[r^{-1}] \rightarrow F$  in  $R\text{-Mod}$  vanishes by the definition of the category  $\mathcal{P}$ , hence  $f = 0$  and consequently  $f_n = 0$  for all  $n$ .

Similarly one can show that the zero object is also a transfinitely iterated extension of the sequence of objects  $P$ ,  $S$ ,  $S$ ,  $S$ ,  $\dots$  in  $\mathcal{P}$ , where  $P = \lim_m R/r^m R$ . It suffices to set  $F_n = P$  and take  $f_{ij}$  to be the morphisms of multiplication with  $r^{j-i}$ . Then the colimit of the diagram  $(f_{ij} : F_i \rightarrow F_j)$  in  $R\text{-Mod}$  is the  $R[r^{-1}]$ -module  $R[r^{-1}] \otimes_R P$ , whose reflection to  $\mathcal{P}$  vanishes, so  $\text{colim}_n F_n = 0$  in  $\mathcal{P}$ .

These counterexamples explain why Eklof and Trlifaj's proof of their theorem [17, Theorems 2 and 10] fails to work in locally presentable abelian categories in general. The following lemma says, on the other hand, that the assertion of Eklof's lemma [17, Lemma 1] is valid for any transfinitely iterated extension of objects in a cocomplete abelian category.

**Lemma 4.5.** *Let  $\mathcal{K}$  be a cocomplete abelian category and  $\mathcal{B}$  a class of objects in  $\mathcal{K}$ . Then the class of objects  ${}^\perp \mathcal{B} \subseteq \mathcal{K}$  is closed under transfinitely iterated extensions.*

*Proof.* This can be easily obtained from the lifting property considerations, but we will outline a sketch of explicit proof directly comparable to the argument in [17]. Let  $F$  be a transfinitely iterated extension of a family of objects  $S_i \in {}^\perp \mathcal{B}$ ,  $i < \alpha$ , and let

$$0 \longrightarrow B \longrightarrow K \xrightarrow{p} F \longrightarrow 0$$

be a short exact sequence in  $\mathcal{K}$  with  $B \in \mathcal{B}$ . In order to show that this exact sequence splits, we proceed by transfinite induction constructing a compatible cocone  $k_i : F_i \rightarrow K$  such that  $pk_i = f_{i\alpha}$  for all  $i \leq \alpha$ . On a limit step  $j$ , we simply set  $k_j = \text{colim}_{i < j} k_i$ . On a successor step  $j = i + 1$ , the pullback of our short exact sequence with respect to the morphism  $f_{i+1,\alpha} : F_{i+1} \rightarrow F$  defines an extension class in  $\text{Ext}_{\mathcal{K}}^1(F_{i+1}, B)$ . The argument continues exactly as in [17, Lemma 1], proving that this extension class vanishes, that there exists a morphism  $k'_{i+1} : F_{i+1} \rightarrow K$  for which  $pk'_{i+1} = f_{i+1,\alpha}$ , and finally that the morphism  $k'_{i+1}$  can be replaced with another morphism  $k_{i+1} : F_{i+1} \rightarrow K$  for which  $pk_{i+1} = f_{i+1,\alpha}$  and  $k_{i+1}f_{i,i+1} = k_i$ .  $\square$



Given a class of objects  $\mathcal{S}$  in a cocomplete abelian category  $\mathcal{K}$ , we denote by  $\text{filt}(\mathcal{S})$  the class of all transfinitely iterated extensions of objects from  $\mathcal{S}$ . Given a class of morphisms  $\mathcal{L}$  in  $\mathcal{K}$ , we denote by  $\text{cell}(\mathcal{L})$  the closure of  $\mathcal{L}$  with respect to pushouts and transfinite compositions, or, which is the same, the class of all transfinite compositions of pushouts of morphisms from  $\mathcal{L}$ . Clearly, any pushout of an  $\mathcal{S}$ -monomorphism is an  $\mathcal{S}$ -monomorphism; so the class  $\text{cell}(\mathcal{S}\text{-Mono})$  consists precisely of the transfinite compositions of  $\mathcal{S}$ -monomorphisms.

The next lemma is our version of [37, Lemma 2.10 and Proposition 2.11] (see also [40, Appendix A]).

**Lemma 4.6.** *Let  $\mathcal{K}$  be a cocomplete abelian category and  $\mathcal{S}$  a class of objects in  $\mathcal{K}$ . Then*

- (a) *any  $\text{filt}(\mathcal{S})$ -monomorphism in  $\mathcal{K}$  belongs to  $\text{cell}(\mathcal{S}\text{-Mono})$ ;*
- (b) *the cokernel of any morphism from  $\text{cell}(\mathcal{S}\text{-Mono})$  belongs to  $\text{filt}(\mathcal{S})$ ;*
- (c) *any morphism from  $\text{cof}(\mathcal{S}\text{-Mono})$  whose domain is a subobject of an object from  $\mathcal{S}^\perp$  is a monomorphism;*
- (d) *the class of objects  $\text{filt}(\text{filt}(\mathcal{S}))$  coincides with  $\text{filt}(\mathcal{S})$ .*

*Proof.* Part (a): let  $A \rightarrow B$  be a monomorphism whose cokernel  $F$  is a transfinitely iterated extension of objects  $S_i \in \mathcal{S}$ ,  $i < \alpha$ . Let  $B_i \rightarrow F_i$  be the pullback of the epimorphism  $B \rightarrow F$  with respect to the morphism  $f_{i\alpha} : F_i \rightarrow F$ . Clearly, we have a commutative diagram  $(b_{ij} : B_i \rightarrow B_j)_{i < j \leq \alpha}$ , the morphism  $b_{i,i+1}$  is a monomorphism with the cokernel  $S_i$ , and it only remains to show that  $(b_{ij})$  is a smooth chain, that is  $B_j = \text{colim}_{i < j} B_i$  for all limit ordinals  $j \leq \alpha$ . For every  $i$ , we have a commutative diagram of a morphism of short exact sequences with the same subobject

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & F \longrightarrow 0 \\ & & & \searrow & \uparrow & & \uparrow \\ & & & & B_i & \longrightarrow & F_i \longrightarrow 0 \end{array}$$

Passing to the colimit, we conclude that the morphism  $\text{colim}_{i < j} B_i \rightarrow \text{colim}_{i < j} F_i = F_j$  is the cokernel of the morphism  $A \rightarrow \text{colim}_{i < j} B_i$ , since colimits commute with cokernels. Furthermore,  $A \rightarrow \text{colim}_{i < j} B_i$  is a monomorphism, because the composition  $A \rightarrow \text{colim}_{i < j} B_i \rightarrow B$  is. Hence we obtain a commutative diagram of a morphism between two short exact sequences with same subobject and quotient object

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & \text{colim}_{i < j} B_i & \longrightarrow & F_j \longrightarrow 0 \\ & & \searrow & & \downarrow & \nearrow & \\ & & & & B_j & & \end{array}$$

implying that  $\text{colim}_{i < j} B_i \rightarrow B_j$  is an isomorphism.

Part (b) is a particular case of the assertion that a pushout of a transfinite composition is a transfinite composition of pushouts, following from the fact that colimits

commute with pushouts. To prove (d), one notices that, according to part (a),

$$\text{filt}(\text{filt}(\mathcal{S}))\text{-Mono} \subseteq \text{cell}(\text{filt}(\mathcal{S})\text{-Mono}) \subseteq \text{cell}(\text{cell}(\mathcal{S}\text{-Mono})) = \text{cell}(\mathcal{S}\text{-Mono}),$$

so the inclusion  $\text{filt}(\text{filt}(\mathcal{S})) \subseteq \text{filt}(\mathcal{S})$  follows from (b). Finally, part (c) is implied by the dual of Lemma 4.2(c), since morphisms from  $\text{cof}(\mathcal{S}\text{-Mono})$  have the left lifting property with respect to morphisms from  $\mathcal{S}^\perp\text{-Epi}$  by Lemma 4.2(a).  $\square$

**Remark 4.7.** Starting from Section 3, we always assumed the categories where we considered transfinite compositions of morphisms or transfinitely iterated extensions of objects to be cocomplete, just to help the reader feel more confident; but this assumption was never actually used until now. It was always sufficient to say instead that the classes  $\text{cof}(\mathcal{L})$ ,  $\text{cell}(\mathcal{L})$ , and  $\text{filt}(\mathcal{S})$  are formed using all the colimits that happen to exist in the category  $\mathcal{K}$ .

The above proof of Lemma 4.6(a), however, presumes existence of  $\text{colim}_{i < j} B_i$ . This can be avoided, too. Here is an alternative argument proving directly that  $(b_{ij} : B_i \rightarrow B_j)_{i < j}$  is a colimit cocone without any prior assumption of the existence of the colimit. We have to show that for any object  $X$  in  $\mathcal{K}$  the induced morphism of abelian groups  $c_X : \mathcal{K}(B_j, X) \rightarrow \lim_{i < j} \mathcal{K}(B_i, X)$  is an isomorphism. We have the commutative diagram of a morphism of exact sequences of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & \lim_{i < j} \mathcal{K}(F_j, X) & \longrightarrow & \lim_{i < j} \mathcal{K}(B_i, X) & \longrightarrow & \mathcal{K}(A, X) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathcal{K}(F_j, X) & \longrightarrow & \mathcal{K}(B_j, X) & \longrightarrow & \mathcal{K}(A, X) \end{array}$$

The leftmost and the rightmost vertical maps are isomorphisms, hence it follows from the diagram that the middle vertical map  $c_X$  is injective and a compatible cocone  $(x_i : B_i \rightarrow X)_{i < j}$  belongs to the image of  $c_X$ , i. e., factors through the cocone  $(b_{ij})_{i < j}$ , if and only if the morphism  $x_0 : A \rightarrow X$  factors through  $A \rightarrow B_j$ .

Now let  $h : X \rightarrow Y$  be the pushout of the monomorphism  $A \rightarrow B_j$  by the morphism  $x_0$ . Then the composition  $hx_0 : A \rightarrow Y$  factors through  $A \rightarrow B_j$ , hence the cocone  $(hx_i : B_i \rightarrow Y)_{i < j}$  belongs to the image of  $c_Y$ . We have obtained a morphism  $y : B_j \rightarrow Y$  forming a commutative diagram with the morphisms  $hx_i$  and  $b_{ij}$ . Let  $p : Y \rightarrow Z$  be the cokernel of the monomorphism  $h$ . Then  $c_Z(py) = (phx_i)_{i < j}$  is the zero cocone, and since the morphism  $c_Z$  is injective, it follows that  $py = 0$ . Thus there is a morphism  $x : B_j \rightarrow X$  for which  $y = hx$ , and one has  $xb_{ij} = x_i$ , since  $hxb_{ij} = yb_{ij} = hx_i$ . We have proven that the map  $c_X$  is also surjective.

The following theorem is a more precise formulation of the Eklof–Trlifaj theorem for locally presentable abelian categories (cf. Corollary 3.6; see also Proposition 4.16).

**Theorem 4.8.** *Let  $\mathcal{K}$  be a locally presentable abelian category,  $\mathcal{S}$  a set of objects in  $\mathcal{K}$ , and  $(\mathcal{F}, \mathcal{C})$  the cotorsion theory in  $\mathcal{K}$  generated by  $\mathcal{S}$ . In this situation:*

- (a) *the class of objects  $\text{filt}(\mathcal{S})$  is contained in  $\mathcal{F}$ ;*

(b) if an object  $X$  in  $\mathcal{K}$  is a subobject of an object  $B$  from the class  $\mathcal{C}$ , then  $X$  can be included into a short exact sequence  $0 \rightarrow X \rightarrow C \rightarrow F' \rightarrow 0$  in  $\mathcal{K}$  with  $C \in \mathcal{C}$  and  $F' \in \text{filt}(\mathcal{S})$ ;

(c) if an object  $Y$  in  $\mathcal{K}$  is a quotient of an object  $P$  from  $\text{filt}(\mathcal{S})$ , then  $Y$  can be included into a short exact sequence  $0 \rightarrow C' \rightarrow F \rightarrow Y \rightarrow 0$  in  $\mathcal{K}$  with  $C' \in \mathcal{C}$  and  $F \in \text{filt}(\mathcal{S})$ ;

(d) if an object  $G$  from the class  $\mathcal{F}$  is a quotient object of an object from  $\text{filt}(\mathcal{S})$ , then  $G$  is a direct summand of an object from  $\text{filt}(\mathcal{S})$ .

*Proof.* Part (a) is Lemma 4.5. To prove parts (b–d), choose a regular cardinal  $\lambda$  such that the category  $\mathcal{K}$  is locally  $\lambda$ -presentable and all objects in the set  $\mathcal{S}$  are  $\lambda$ -presentable. Let  $\mathcal{I}$  be the set of all  $\mathcal{S}$ -monomorphisms with  $\lambda$ -presentable codomains in  $\mathcal{K}$ . By Lemmas 3.4 and 4.6(a), we have  $\mathcal{I}^\square = \mathcal{S}\text{-Mono}^\square = \text{filt}(\mathcal{S})\text{-Mono}^\square$ .

According to Lemma 4.2, the class  $\mathcal{S}\text{-Mono}^\square$  consists of morphisms with kernels in  $\mathcal{C}$ , contains all the  $\mathcal{C}$ -epimorphisms, and any morphism from  $\mathcal{S}\text{-Mono}^\square$  whose codomain is a quotient of an object from  $\text{filt}(\mathcal{S})$  is an epimorphism. According to the dual of Lemma 4.2, the class  ${}^\square(\mathcal{S}\text{-Mono}^\square)$  consists of morphisms with cokernels in  $\mathcal{F}$  and any morphism from  ${}^\square(\mathcal{S}\text{-Mono}^\square)$  whose domain is a subobject of an object from  $\mathcal{C}$  is a monomorphism. Moreover, by Lemma 4.6(b) the class of morphisms  $\text{cell}(\mathcal{S}\text{-Mono}) \subset {}^\square(\mathcal{S}\text{-Mono}^\square)$  consists of morphisms with cokernels in  $\text{filt}(\mathcal{S})$ .

By the small object argument (see [5, Proposition 1.3] or [32, Theorem 2.5]), any morphism in  $\mathcal{K}$  can be factored as a composition of a morphism from  $\text{cell}(\mathcal{S}\text{-Mono})$  followed by a morphism from  $\mathcal{S}\text{-Mono}^\square$ . This implies the assertions (b–c), and part (d) follows from (c).  $\square$

A class of objects  $\mathcal{F}$  in a cocomplete abelian category  $\mathcal{K}$  is called *deconstructible* if  $\mathcal{F} = \text{filt}(\mathcal{S})$  for some set of objects  $\mathcal{S}$  in  $\mathcal{K}$ . The next corollary is our version of [37, Theorem 2.13(1) and Corollary 2.15(1)].

**Corollary 4.9.** *Any deconstructible class of objects in a locally presentable abelian category is weakly coreflective.*

*Proof.* For any class of objects  $\mathcal{A}$  in an abelian category  $\mathcal{K}$ , any morphism belonging to  $\mathcal{A}\text{-Mono}^\square$  with the domain belonging to  $\mathcal{A}$  is an  $\mathcal{A}$ -precover. In particular, let  $\mathcal{K}$  be locally presentable and  $\mathcal{S}$  a set of objects in  $\mathcal{K}$ . Following the proof of Theorem 4.8, for any object  $K$  in  $\mathcal{K}$  the morphism  $0 \rightarrow K$  can be decomposed as  $0 \rightarrow F \rightarrow K$  with the object  $F$  from  $\text{filt}(\mathcal{S})$  and the morphism  $F \rightarrow K$  from  $\mathcal{S}\text{-Mono}^\square = \text{filt}(\mathcal{S})\text{-Mono}^\square$ . Hence  $F \rightarrow K$  is a  $\text{filt}(\mathcal{S})$ -precover of  $K$ .  $\square$

**Remark 4.10.** According to [23, Example 4], for any cotorsion theory  $(\mathcal{F}, \mathcal{C})$  generated by a set of objects in a locally presentable abelian category  $\mathcal{K}$ , any object  $X$  in  $\mathcal{K}$  has a  $\mathcal{C}$ -preenvelope. If one of such preenvelopes is a monomorphism (or, equivalently, all of them are), then by Theorem 4.8(b)  $X$  has a special  $\mathcal{C}$ -preenvelope.

Assuming Vopěnka's principle, any full subcategory closed under products in a locally presentable category is weakly reflective (see [1, Theorem 6.26]).

**Corollary 4.11.** *Let  $\mathcal{K}$  be a locally presentable abelian category and  $(\mathcal{F}, \mathcal{C})$  a cotorsion theory generated by a set such that  $\mathcal{F}$  is closed under  $\kappa$ -directed colimits for some regular cardinal  $\kappa$ . Suppose that any object of  $\mathcal{K}$  is a quotient of an object from  $\mathcal{F}$ . Then  $\mathcal{F}$  is accessible.*

*Proof.* According to the proof of Proposition 3.5, one can choose a generating set  $\mathcal{S}$  in such a way that any object of  $\mathcal{K}$  is a quotient of a coproduct of objects from  $\mathcal{S}$ . Then, by the proof of Theorem 4.8, an object  $F$  in  $\mathcal{K}$  is  $\mathcal{I}$ -cofibrant, i. e., the morphism  $0 \rightarrow F$  belongs to  $\text{cof}(\mathcal{I})$ , if and only if  $F$  is from  $\mathcal{F}$ . The desired assertion now follows from [26, Corollary 5.1].  $\square$

Assuming Vopěnka's principle, any full subcategory of  $\mathcal{K}$  closed under  $\kappa$ -directed colimits is accessible, and so is any full subcategory of  $\mathcal{K}$  closed under  $\kappa$ -pure subobjects (see [1, Theorem 6.17]).

**Corollary 4.12.** *Assume that there is a proper class of almost strongly compact cardinals. Let  $\mathcal{K}$  be a locally presentable abelian category and  $(\mathcal{F}, \mathcal{C})$  a cotorsion theory generated by a set such that  $\mathcal{F}$  is closed under  $\kappa$ -pure subobjects in  $\mathcal{K}$  for some regular cardinal  $\kappa$ . Suppose that any object of  $\mathcal{K}$  is a quotient of an object from  $\mathcal{F}$ . Then the category  $\mathcal{F}$  is accessible and accessibly embedded into  $\mathcal{K}$ .*

*Proof.* Let  $\lambda \geq \kappa$  be a regular cardinal such that  $\mathcal{K}$  is locally  $\lambda$ -presentable and there is a set of morphisms  $\mathcal{I}$  with  $\lambda$ -presentable domains and codomains in  $\mathcal{K}$  such that  $\mathcal{F}$  is the class of  $\mathcal{I}$ -cofibrant objects. Following [34, Proposition 3.4], there is a  $\lambda$ -accessible functor  $F : \mathcal{K} \rightarrow \mathcal{K}$  such that the class  $\mathcal{F}$  consists precisely of the retracts of objects in the image of  $F$ . Since  $\mathcal{F}$  is closed under  $\kappa$ -pure subobjects, we can conclude that  $\mathcal{F}$  is the  $\lambda$ -pure powerful image of  $F$ . According to [10, Proposition 2.4, Theorem 3.2, and Remark 3.3],  $\mathcal{F}$  is accessible and accessibly embedded.  $\square$

**Remark 4.13.** Following the proof of Theorem 4.8 more closely, in both Corollaries 4.11 and 4.12 one can drop the assumption that every object of  $\mathcal{K}$  is a quotient of an object of  $\mathcal{F}$ , replacing in their assertions the left part  $\mathcal{F}$  of a cotorsion theory  $(\mathcal{F}, \mathcal{C})$  with the class  ${}^\oplus \text{filt}(\mathcal{S})$  of all direct summands of objects from  $\text{filt}(\mathcal{S})$ , where  $\mathcal{S}$  is any set of objects in  $\mathcal{K}$ . So, in the case of Corollary 4.11, it is then claimed that the full subcategory  ${}^\oplus \text{filt}(\mathcal{S})$  is accessible whenever it is closed under  $\kappa$ -directed colimits in  $\mathcal{K}$ . In the case of Corollary 4.12 the assertion is that, assuming a proper class of almost strongly compact cardinals,  ${}^\oplus \text{filt}(\mathcal{S})$  is accessible and accessibly embedded whenever it is closed under  $\kappa$ -pure subobjects.

The next lemma is our version of [6, Lemma 1 and Proposition 2].

**Lemma 4.14.** *Let  $\mathcal{K}$  be a locally presentable abelian category and  $\mathcal{F}$  a class of objects in  $\mathcal{K}$  closed under transfinitely iterated extensions,  $\kappa$ -pure subobjects, and  $\kappa$ -pure quotients for some regular cardinal  $\kappa$ . Assume that the class of all  $\mathcal{F}$ -monomorphisms*

$X \rightarrow F$  is closed under directed colimits in the comma-category  $\mathcal{K} \downarrow F$  for any  $F \in \mathcal{F}$ . Then the class  $\mathcal{F}$  is deconstructible.

*Proof.* Let  $\lambda \geq \kappa$  be a regular cardinal for which  $\mathcal{K}$  is locally  $\lambda$ -presentable. Let  $\alpha \geq \lambda$  be a regular cardinal such that any morphism from an  $\alpha$ -presentable object to an object  $K$  in  $\mathcal{K}$  factors through an  $\alpha$ -presentable  $\lambda$ -pure subobject of  $K$  (see [1, Theorem 2.33]). Then we claim that  $\mathcal{F} = \text{filt}(\mathcal{S})$ , where  $\mathcal{S}$  is a representative set of all objects in  $\mathcal{F}$  that are  $\alpha$ -presentable in  $\mathcal{K}$ . Indeed, let an object  $F \in \mathcal{F}$  be  $\mu$ -presentable in  $\mathcal{K}$ , where  $\mu \geq \lambda$  is a regular cardinal greater than the cardinality  $\gamma$  of the set of all morphisms between  $\lambda$ -presentable objects in  $\mathcal{K}$ . We construct a smooth chain  $(f_{ij} : F_i \rightarrow F_j)_{i < j < \mu}$  and a compatible cocone  $(f_i : F_i \rightarrow F)$  such that  $F_i \in \mathcal{F}$ , the morphisms  $f_{i,i+1}$  are  $\mathcal{S}$ -monomorphisms, and  $f_i$  are  $\mathcal{F}$ -monomorphisms.

Set  $F_0 = 0$ . On a successor step  $i + 1$ , if the morphism  $f_i$  is an isomorphism, put  $F_{i+1} = F_i$ ,  $f_{i+1} = f_i$ , and  $f_{i,i+1} = \text{id}$ . Otherwise, the cokernel  $\text{coker}(f_i)$  is nonzero, so there exists a nonzero morphism from a  $\lambda$ -presentable object to  $\text{coker}(f_i)$ , which consequently factors through a nonzero  $\alpha$ -presentable  $\lambda$ -pure subobject  $s_i : S_i \rightarrow \text{coker}(f_i)$ . Since  $\mathcal{F}$  is closed under  $\lambda$ -pure subobjects, we have  $S_i \in \mathcal{S}$ , and since  $\mathcal{F}$  is closed under  $\lambda$ -pure quotients,  $\text{coker}(s_i) \in \mathcal{F}$ . Let  $f_{i+1} : F_{i+1} \rightarrow F$  be the pullback of the morphism  $s_i$  with respect to the epimorphism  $F \rightarrow \text{coker}(f_i)$ . Then  $f_{i+1}$  is a monomorphism with  $\text{coker}(f_{i+1}) = \text{coker}(s_i)$ , there exists a unique morphism  $f_{i,i+1} : F_i \rightarrow F_{i+1}$  with  $f_{i+1}f_{i,i+1} = f_i$ , and  $f_{i,i+1}$  is a monomorphism with  $\text{coker}(f_{i,i+1}) = S_i$ . Since  $F_{i+1}$  is an extension of  $S_i$  and  $F_i$ , and  $\mathcal{F}$  is closed under extensions, we have  $F_{i+1} \in \mathcal{F}$ . On a limit step, we use the assumptions that  $\mathcal{F}$  is closed under transinitely iterated extensions and the class of all  $\mathcal{F}$ -monomorphisms is closed under directed colimits in  $\mathcal{K} \downarrow F$ .

Since  $F$  cannot have an increasing chain of subobjects of length  $\mu$  (see Lemma 2.3), there exists an ordinal  $j < \mu$  such that  $f_j : F_j \rightarrow F$  is an isomorphism. We have proved that  $F$  is a transinitely iterated extension of the objects  $S_i$ ,  $i < j$ .  $\square$

The following lemma and proposition provide the most straightforward extension of the first half of the proof of the Eklof–Trlifaj theorem [17, Theorem 10] to locally presentable abelian categories.

**Lemma 4.15.** *Let  $\mathcal{K}$  be a cocomplete abelian category and  $\mathcal{H}$  a class of objects closed under extensions in  $\mathcal{K}$ . Let  $K$  be an object in  $\mathcal{K}$  such that the class of all  $\mathcal{H}$ -monomorphisms  $K \rightarrow X$  is closed under directed colimits in  $\mathcal{K} \downarrow K$ . Then any morphism  $K \rightarrow X$  belonging to  $\text{cell}(\mathcal{H}\text{-Mono})$  is an  $\mathcal{H}$ -monomorphism.*

*Proof.* It is an immediate corollary of the definitions.  $\square$

**Proposition 4.16.** *Let  $\mathcal{K}$  be a locally presentable abelian category and  $(\mathcal{F}, \mathcal{C})$  the cotorsion theory generated by a set of objects  $\mathcal{S}$  in  $\mathcal{K}$ . Assume that the set  $\mathcal{S}$  is contained in a class of objects  $\mathcal{H}$  closed under extensions and strongly closed under directed colimits in  $\mathcal{K}$ . Then any object of  $\mathcal{K}$  has a special  $\mathcal{C}$ -preenvelope.*

*Proof.* To be more precise, let  $K$  be an object in  $\mathcal{K}$  such that the set  $\mathcal{S}$  is contained in a class of objects  $\mathcal{H}$  closed under extensions in  $\mathcal{K}$  for which the class of all  $\mathcal{H}$ -monomorphisms  $K \rightarrow X$  is closed under directed colimits in  $K \downarrow \mathcal{K}$ . Then any morphism  $K \rightarrow X$  belonging to  $\text{cell}(\mathcal{S}\text{-Mono})$  is a monomorphism by Lemma 4.15. Following the proof of Theorem 4.8, the morphism  $K \rightarrow 0$  can be decomposed as  $K \rightarrow C \rightarrow 0$ , where the morphism  $K \rightarrow C$  belongs to  $\text{cell}(\mathcal{S}\text{-Mono})$  and the object  $C$  belongs to the class  $\mathcal{C}$ . Furthermore, the cokernel of any morphism from  $\text{cell}(\mathcal{S}\text{-Mono})$  belongs to  $\text{filt}(\mathcal{S})$ . We have constructed a short exact sequence  $0 \rightarrow K \rightarrow C \rightarrow F' \rightarrow 0$  with  $C \in \mathcal{C}$  and  $F' \in \text{filt}(\mathcal{S})$ . Finally,  $\text{filt}(\mathcal{S}) \subseteq \mathcal{F}$  by Theorem 4.8(a).  $\square$

To finish this section, let us formulate the somewhat more precise versions of Corollaries 3.7 and 3.10 that the techniques of their proofs allow to obtain.

**Corollary 4.17.** *Let  $\mathcal{K}$  be a locally presentable category and  $\mathcal{A}$  a class of objects in  $\mathcal{K}$  closed under directed colimits. Then whenever an object of  $\mathcal{K}$  that has an  $\mathcal{A}$ -precover, it also has an  $\mathcal{A}$ -cover.*  $\square$

**Corollary 4.18.** *Let  $\mathcal{K}$  be a locally presentable abelian category and  $(\mathcal{F}, \mathcal{C})$  a cotorsion theory in  $\mathcal{K}$ . Let  $K$  be an object of  $\mathcal{K}$  such that the class of all  $\mathcal{F}$ -monomorphisms  $K \rightarrow X$  is closed under directed colimits in the comma-category  $K \downarrow \mathcal{K}$ . Assume that the object  $K$  has a special  $\mathcal{C}$ -preenvelope. Then  $K$  also has a  $\mathcal{C}$ -envelope.*  $\square$

**Remark 4.19.** Weakening the assumptions of Corollary 4.18 a bit further, let  $\mathcal{F}$  be a class of objects closed under extensions in a locally presentable abelian category  $\mathcal{K}$ . Let  $K \in \mathcal{K}$  be an object such that the class of all  $\mathcal{F}$ -monomorphisms  $K \rightarrow X$  is closed under directed colimits in  $K \downarrow \mathcal{K}$ . Set  $\mathcal{C} = \mathcal{F}^\perp$ , and assume that there exists an  $\mathcal{F}$ -monomorphism  $K \rightarrow C$  with  $C \in \mathcal{C}$ . Then the object  $K$  has a  $\mathcal{C}$ -envelope.

**Corollary 4.20.** *Let  $\mathcal{K}$  be a locally presentable abelian category and  $(\mathcal{F}, \mathcal{C})$  a cotorsion theory generated by a set of objects in  $\mathcal{K}$ . Let  $K$  be an object of  $\mathcal{K}$  such that the class of all  $\mathcal{F}$ -monomorphisms  $K \rightarrow X$  is closed under directed colimits in the comma-category  $K \downarrow \mathcal{K}$ . Then  $K$  has a  $\mathcal{C}$ -envelope.*

*Proof.* This follows from Corollary 4.18 and (the proof of) Proposition 4.16.  $\square$

## 5. CONTRAMODULES OVER TOPOLOGICAL RINGS

Let  $\mathfrak{R}$  be an associative, unital topological ring where open right ideals  $\mathfrak{I} \subseteq \mathfrak{R}$  form a base of neighborhoods of zero. We will always assume that  $\mathfrak{R}$  is complete and separated, that is the natural map  $\mathfrak{R} \rightarrow \lim_{\mathfrak{I}} \mathfrak{R}/\mathfrak{I}$  from  $\mathfrak{R}$  to the limit of its quotients by its open right ideals is an isomorphism (of abelian groups, or of right  $\mathfrak{R}$ -modules).

For any topological abelian group  $\mathfrak{A}$  where open subgroups  $\mathfrak{U} \subseteq \mathfrak{A}$  form a base of neighborhoods of zero and any set  $X$ , we denote by  $\mathfrak{A}[[X]] \subseteq \mathfrak{A}^X$  the set of all infinite formal linear combinations  $\sum_x a_x x$  of elements  $x$  of  $X$  with families of coefficients  $a_x \in \mathfrak{A}$  converging to zero in the topology of  $\mathfrak{A}$ . The latter condition means that for any open subgroup  $\mathfrak{U} \subseteq \mathfrak{A}$  the set of all  $x \in X$  for which  $a_x \notin \mathfrak{U}$  must be finite.

When the topological group  $\mathfrak{A}$  is complete and separated, that is  $\mathfrak{A} = \lim_{\mathfrak{U}} \mathfrak{A}/\mathfrak{U}$ , one has  $\mathfrak{A}[[X]] = \lim_{\mathfrak{U}} \mathfrak{A}/\mathfrak{U}[X]$ , where  $A[X] = A^{(X)}$  denotes the set of all finite linear combinations of elements of  $X$  with coefficients in a group  $A$ . Given a map of sets  $f : X \rightarrow Y$ , there is the induced “direct image” map  $\mathfrak{A}[[f]] : \mathfrak{A}[[X]] \rightarrow \mathfrak{A}[[Y]]$  taking  $\sum_x a_x x$  to  $\sum_y (\sum_{f(x)=y} a_x) y$ , where the sum of the coefficients  $a_x \in \mathfrak{A}$  over all the preimages  $x \in X$  of a given element  $y \in Y$  is defined as the limit of finite partial sums converging in the topology of  $\mathfrak{A}$ .

For any associative, unital ring  $R$  and any set  $X$  there is a natural map of “opening of parentheses”  $\phi_R(X) : R[R[X]] \rightarrow R[X]$  defining, together with the “point measure” map  $\epsilon_R(X) : X \rightarrow R[X]$ , the structure of a monad on the category of sets on the functor  $X \mapsto R[X]$ . The category of left  $R$ -modules is equivalent to the category of algebras/modules over this monad on the category of sets.

Similarly, for any topological ring  $\mathfrak{R}$  with the topology of the kind described above and for any set  $X$  there is an “opening of parentheses” map  $\phi_{\mathfrak{R}}(X) : \mathfrak{R}[[\mathfrak{R}[[X]]]] \rightarrow \mathfrak{R}[[X]]$  involving infinite summation of coefficients understood as the limit of finite partial sums in  $\mathfrak{R}$  (our conditions on the topology of  $\mathfrak{R}$  are designed to ensure the convergence). More specifically, in order to obtain the map  $\phi_{\mathfrak{R}}(X)$  it suffices to construct a morphism of diagrams  $\mathfrak{R}/\mathfrak{I}[\mathfrak{R}[[X]]] \rightarrow \mathfrak{R}/\mathfrak{I}[X]$  indexed by the partially ordered set of all open right ideals  $\mathfrak{I} \subseteq \mathfrak{R}$ . This is easily done using the fact that for any element  $r \in \mathfrak{R}$  and any open right ideal  $\mathfrak{I}$  in  $\mathfrak{R}$  there exists an open right ideal  $\mathfrak{I}'$  in  $\mathfrak{R}$  such that  $r\mathfrak{I}' \subseteq \mathfrak{I}$ . The natural transformation  $\phi_{\mathfrak{R}}(X)$ , together with the “point measure” map  $\epsilon_{\mathfrak{R}}(X) : X \rightarrow \mathfrak{R}[[X]]$ , define the structure of a monad on the category of sets on the functor  $X \mapsto \mathfrak{R}[[X]]$ .

**Definition 5.1.** The category of *left  $\mathfrak{R}$ -contramodules*  $\mathfrak{R}\text{-Contra}$  is defined as the category of algebras/modules over the monad  $\mathfrak{R}[[ - ]] : \mathbf{Set} \rightarrow \mathbf{Set}$ . In other words, a left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$  is a set endowed with a *left contraaction* map  $\pi_{\mathfrak{P}} : \mathfrak{R}[[\mathfrak{P}]] \rightarrow \mathfrak{P}$  satisfying the equations of *contraassociativity*  $\pi_{\mathfrak{P}} \circ \phi_{\mathfrak{R}}(\mathfrak{P}) = \pi_{\mathfrak{P}} \circ \mathfrak{R}[[\pi_{\mathfrak{P}}]]$

$$\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$$

and *contraunitality*  $\pi_{\mathfrak{P}} \circ \epsilon_{\mathfrak{R}}(\mathfrak{P}) = \text{id}_{\mathfrak{P}}$

$$\mathfrak{P} \longrightarrow \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}.$$

One can view an  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$  as a set endowed with *infinite summation operations* assigning to any set  $X$ , an  $X$ -indexed family of elements  $r_x \in \mathfrak{R}$  converging to zero in the topology of  $\mathfrak{R}$ , and an  $X$ -indexed family of elements  $p_x \in \mathfrak{P}$  an element  $\pi_{\mathfrak{P}}(\sum_x r_x p_x) \in \mathfrak{P}$ . Here  $\sum_x r_x p_x \in \mathfrak{R}[[\mathfrak{P}]]$  is a notation for the image of the element  $\sum_x r_x x \in \mathfrak{R}[[X]]$  with respect to the “direct image” map  $\mathfrak{R}[[X]] \rightarrow \mathfrak{R}[[\mathfrak{P}]]$  induced by the map  $X \rightarrow \mathfrak{P}$  taking  $x$  to  $p_x$ . Composing the contraaction map  $\pi_{\mathfrak{P}}$  with the inclusion  $\mathfrak{R}[\mathfrak{P}] \rightarrow \mathfrak{R}[[\mathfrak{P}]]$  (i. e., restricting oneself to summation operations with finite coefficient families), one constructs the underlying left  $\mathfrak{R}$ -module structure on a left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ . This provides the forgetful functor  $\mathfrak{R}\text{-Contra} \rightarrow \mathfrak{R}\text{-Mod}$ .

The *free  $\mathfrak{R}$ -contramodule* generated by a set  $X$  is the free algebra/module over the monad  $\mathfrak{R}[[ - ]]$  generated by  $X$ , that is the left  $\mathfrak{R}$ -contramodule  $\mathfrak{R}[[X]]$  with the contraaction map  $\pi_{\mathfrak{R}[[X]]} = \phi_{\mathfrak{R}}(X)$ . The free  $\mathfrak{R}$ -contramodule  $\mathfrak{R}[[X]]$  is the coproduct of  $X$  copies of the free  $\mathfrak{R}$ -contramodule with one generator  $\mathfrak{R} = \mathfrak{R}[[\{*\}]]$  in the category of left  $\mathfrak{R}$ -contramodules. The forgetful functor  $\mathfrak{R}\text{-Contra} \rightarrow \mathfrak{R}\text{-Mod}$  can be also constructed as the corepresentable functor  $\mathfrak{R}\text{-Contra}(\mathfrak{R}, -)$ .

Let  $\lambda$  be the successor cardinal of the cardinality of a base of neighborhoods of zero in  $\mathfrak{R}$ . Then the category of left  $\mathfrak{R}$ -contramodules  $\mathfrak{R}\text{-Contra}$  is a locally  $\lambda$ -presentable abelian category (because the cardinality of the support  $\{x \mid r_x \neq 0\}$  of any family of coefficients  $r_x \in \mathfrak{R}$  converging to zero in the topology of  $\mathfrak{R}$  is smaller than  $\lambda$ ). The free  $\mathfrak{R}$ -contramodule with one generator  $\mathfrak{R}$  is a  $\lambda$ -presentable projective generator of  $\mathfrak{R}\text{-Contra}$ . A left  $\mathfrak{R}$ -contramodule is projective if and only if it is a direct summand of the free  $\mathfrak{R}$ -contramodule  $\mathfrak{R}[[X]]$  for some set  $X$ . The forgetful functor  $\mathfrak{R}\text{-Contra} \rightarrow \mathfrak{R}\text{-Mod}$  is exact and preserves arbitrary products (i. e., preserves finite colimits and all limits). It also preserves  $\lambda$ -filtered colimits.

**Example 5.2.** Let  $R$  be a Noetherian commutative ring and  $I \subset R$  an ideal. Consider the  $I$ -adic completion  $\mathfrak{R} = \lim_m R/I^m$  of the ring  $R$ , viewed as a topological ring in the  $I$ -adic topology (or, which is the same, the topology of limit of discrete rings  $R/I^m$ ). Then the forgetful functor  $\mathfrak{R}\text{-Contra} \rightarrow R\text{-Mod}$  is fully faithful. Its image is the full subcategory  $\mathcal{P} \subseteq R\text{-Mod}$  of  $I$ -contramodule  $R$ -modules described in Example 4.1(3) (see [28, Theorem B.1.1]).

Classical counterexamples [30, Section 1.5] show that the contramodule infinite summation cannot be understood as any kind of limit of finite partial sums, as the image of an infinite formal linear combination  $\sum_x r_x p_x \in \mathfrak{R}[[\mathfrak{P}]]$  under the contraaction map  $\pi_{\mathfrak{P}} : \mathfrak{R}[[\mathfrak{P}]] \rightarrow \mathfrak{P}$  may well be nonzero while all the products  $r_x \cdot p_x \in \mathfrak{P}$ , taken in the underlying  $\mathfrak{R}$ -module structure of the  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ , vanish. In fact, the  $I$ -adic topology on an  $I$ -contramodule  $R$ -module  $Q \in \mathcal{P} \subseteq R\text{-Mod}$  does not need to be separated, i. e., the intersection  $\cap_m I^m Q \subseteq Q$  may be nonzero (even though  $IQ = Q$  implies  $Q = 0$  [30, part (b) of Lemma in Section 2.1]).

Let, as above,  $\mathfrak{R}$  be an associative and unital, complete and separated topological ring where open right ideals form a base of neighborhoods of zero.

**Definition 5.3.** A right  $\mathfrak{R}$ -module  $N$  is called *discrete* if the multiplication map  $N \times \mathfrak{R} \rightarrow N$  is continuous in the given topology of  $\mathfrak{R}$  and the discrete topology of  $N$ . Equivalently, this means that for any element  $b \in N$  there exists an open right ideal  $\mathfrak{J} \subseteq \mathfrak{R}$  for which  $b\mathfrak{J} = 0$  in  $N$ . We denote the full subcategory of discrete right  $\mathfrak{R}$ -modules in the category of right  $\mathfrak{R}$ -modules by  $\text{Discr-}\mathfrak{R} \subseteq \text{Mod-}\mathfrak{R}$ .

The full subcategory of discrete right  $\mathfrak{R}$ -modules  $\text{Discr-}\mathfrak{R}$  is closed under all colimits, finite limits, and extensions in  $\text{Mod-}\mathfrak{R}$ . The category  $\text{Discr-}\mathfrak{R}$  is a Grothendieck abelian category and a coreflective subcategory in  $\text{Mod-}\mathfrak{R}$ .



For any abelian group  $A$  and discrete right  $\mathfrak{R}$ -module  $N$ , there is a natural structure of left  $\mathfrak{R}$ -contramodule on the set of all abelian group homomorphisms  $\mathbf{Ab}(N, A)$ . Given a set  $X$ , an  $X$ -indexed family of coefficients  $r_x \in \mathfrak{R}$  converging to zero in the topology of  $\mathfrak{R}$ , and an  $X$ -indexed family of homomorphisms  $f_x \in \mathbf{Ab}(N, A)$ , one defines the homomorphism  $f = \pi_{\mathbf{Ab}(N, A)}(\sum_x r_x f_x)$  by the rule  $f(b) = \sum_x f_x(br_x) \in A$  for all  $b \in N$ . Since the right  $\mathfrak{R}$ -module  $N$  is discrete, one has  $br_x = 0$  for all but a finite subset of indices  $x \in X$ , so the sum is actually finite.

**Definition 5.4.** The *contratensor product*  $N \odot_{\mathfrak{R}} \mathfrak{P}$  of a discrete right  $\mathfrak{R}$ -module  $N$  and a left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$  is an abelian group constructed as the quotient group of the tensor product group over the ring of integers  $N \otimes_{\mathbb{Z}} \mathfrak{P}$  by the subgroup generated by all elements of the form  $b \otimes \pi_{\mathfrak{P}}(\sum_x r_x p_x) - \sum_x br_x \otimes p_x$ , where  $X$  is a set,  $r_x \in \mathfrak{R}$  is an  $X$ -indexed family of elements converging to zero in  $\mathfrak{R}$ , and  $p_x$  is an  $X$ -indexed family of elements in  $\mathfrak{P}$ . One has  $br_x = 0$  for all but a finite subset of indices  $x$ , so the expression in the right-hand side is well-defined.

For any discrete right  $\mathfrak{R}$ -module  $N$ , left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ , and abelian group  $A$ , there is a natural isomorphism of abelian groups

$$\mathbf{Ab}(N \odot_{\mathfrak{R}} \mathfrak{P}, A) \simeq \mathfrak{R}\text{-Contra}(\mathfrak{P}, \mathbf{Ab}(N, A)).$$

In other words, the functor of contratensor product  $N \odot_{\mathfrak{R}} - : \mathfrak{R}\text{-Contra} \rightarrow \mathbf{Ab}$  is left adjoint to the functor of homomorphisms  $\mathbf{Ab}(N, -) : \mathbf{Ab} \rightarrow \mathfrak{R}\text{-Contra}$ .

Given an open right ideal  $\mathfrak{I} \subseteq \mathfrak{R}$  and a left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ , we denote by  $\mathfrak{I} \ltimes \mathfrak{P}$  the image of the subgroup  $\mathfrak{I}[[\mathfrak{P}]] \subseteq \mathfrak{R}[[\mathfrak{P}]]$  under the contraaction map  $\mathfrak{R}[[\mathfrak{P}]] \rightarrow \mathfrak{P}$ . As usually, given a left  $\mathfrak{R}$ -module  $M$ , the notation  $\mathfrak{I}M$  stands for the subgroup of  $M$  generated by elements of the form  $rm$ , where  $r \in \mathfrak{I}$  and  $m \in M$ . So for any left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$  one has  $\mathfrak{I}\mathfrak{P} \subseteq \mathfrak{I} \ltimes \mathfrak{P} \subseteq \mathfrak{P}$ . The quotient  $\mathfrak{R}/\mathfrak{I}$  is a discrete right  $\mathfrak{R}$ -module, and there are natural isomorphisms of abelian groups  $(\mathfrak{R}/\mathfrak{I}) \otimes_{\mathfrak{R}} M = M/\mathfrak{I}M$  and  $(\mathfrak{R}/\mathfrak{I}) \odot_{\mathfrak{R}} \mathfrak{P} = \mathfrak{P}/(\mathfrak{I} \ltimes \mathfrak{P})$ .

**Proposition 5.5.** *Let  $\lambda$  be the successor cardinal of the cardinality of a base of neighborhoods of zero in  $\mathfrak{R}$ . Let  $f : \mathfrak{Q} \rightarrow \mathfrak{P}$  be a  $\lambda$ -pure monomorphism in the category of left  $\mathfrak{R}$ -contramodules. Then for any discrete right  $\mathfrak{R}$ -module  $N$  the induced morphism of the contratensor products*

$$N \odot_{\mathfrak{R}} f : N \odot_{\mathfrak{R}} \mathfrak{Q} \longrightarrow N \odot_{\mathfrak{R}} \mathfrak{P}$$

*is a monomorphism of abelian groups.*

*Proof.* Let  $a_1 \otimes q_1 + \dots + a_n \otimes q_n$  be an element of  $N \otimes_{\mathbb{Z}} \mathfrak{Q}$  whose image in  $N \odot_{\mathfrak{R}} \mathfrak{Q}$  is annihilated by the morphism  $N \odot_{\mathfrak{R}} f$ . Then there exist elements  $b_1, \dots, b_m \in N$ , index sets  $X_1, \dots, X_m$ , families of elements  $r_{k,x} \in \mathfrak{R}$ ,  $x \in X_k$  converging to zero in the topology of  $\mathfrak{R}$ , and families of elements  $p_{k,x} \in \mathfrak{P}$ ,  $x \in X_k$  such that

$$\sum_{j=1}^n a_j \otimes f(q_j) = \sum_{k=1}^m b_k \otimes \pi_{\mathfrak{P}} \left( \sum_{x \in X_k} r_{k,x} p_{k,x} \right) - \sum_{k=1}^m \sum_{x \in X_k} b_k r_{k,x} \otimes p_{k,x}$$

in  $N \otimes_{\mathbb{Z}} \mathfrak{P}$ . Replacing the set  $X_k$  with its subset consisting of all the indices  $x$  for which  $r_{k,x} \neq 0$ , we may assume without loss of generality that the cardinality of  $X_k$  is smaller than  $\lambda$ . Set  $p_k = \pi_{\mathfrak{P}}(\sum_{x \in X_k} r_{k,x} p_{k,x}) \in \mathfrak{P}$ , denote by  $Z_k$  the (finite) set of all  $z \in X_k$  for which  $b_k r_{k,z} \neq 0$ , and put  $b_{k,z} = b_k r_{k,z}$  for every  $z \in Z_k$ . Then the following relation depending only on the additive group structures of  $N$  and  $\mathfrak{P}$

$$\sum_{j=1}^n a_j \otimes f(q_j) = \sum_{k=1}^m b_k \otimes p_k - \sum_{k=1}^m \sum_{z \in Z_k} b_{k,z} \otimes p_{k,z}$$

holds in  $N \otimes_{\mathbb{Z}} \mathfrak{P}$ . Hence there exist a free abelian group  $T$  with a finite set of generators  $t_1, \dots, t_l$ , elements  $q'_j, p'_k, p'_{k,z}$  in  $T$ , where  $z \in Z_k$ , a subgroup  $S \subset T$  generated by some elements  $s_1, \dots, s_e$ , and an abelian group homomorphism  $t' : T \rightarrow \mathfrak{P}$  such that  $t'(q'_j) = f(q_j)$ ,  $t'(p'_k) = p_k$ ,  $t'(p'_{k,z}) = p_{k,z}$ ,  $t'(S) = 0$ , and the relation

$$\sum_{j=1}^n a_j \otimes q'_j = \sum_{k=1}^m b_k \otimes p'_k - \sum_{k=1}^m \sum_{z \in Z_k} b_{k,z} \otimes p'_{k,z}$$

holds in  $N \otimes_{\mathbb{Z}} (T/S)$ .

Now let  $\mathfrak{T}$  denote the free left  $\mathfrak{R}$ -contramodule generated by the symbols  $t'_1, \dots, t'_l$  and  $p''_{k,y}$ ,  $y \in X_k \setminus Z_k$ . Denote by  $t'' : T \rightarrow \mathfrak{T}$  the abelian group homomorphism taking  $t_i$  to  $t'_i$ ,  $1 \leq i \leq l$ , and set  $q''_j = t''(q'_j)$ ,  $p''_k = t''(p'_k)$ ,  $s''_i = t''(s_i)$ , and  $p''_{k,z} = t''(p'_{k,z}) \in \mathfrak{T}$  for  $z \in Z_k$ . Let  $\mathfrak{S}$  be the free left  $\mathfrak{R}$ -contramodule generated by the symbols  $s'_1, \dots, s'_e$  and  $w'_1, \dots, w'_m$ . Denote by  $s : \mathfrak{S} \rightarrow \mathfrak{T}$  the  $\mathfrak{R}$ -contramodule morphism taking  $s'_i$  to  $s''_i$ ,  $1 \leq i \leq e$ , and  $w'_k$  to  $p''_k - \pi_{\mathfrak{T}}(\sum_{x \in X_k} r_{k,x} p''_{k,x})$ ,  $1 \leq k \leq m$ . Let  $t : \mathfrak{T} \rightarrow \mathfrak{P}$  be the  $\mathfrak{R}$ -contramodule morphism taking  $t'_i$  to  $t'(t_i)$ ,  $1 \leq i \leq l$ , and  $p''_{k,y}$  to  $p_{k,y}$ ,  $y \in X_k \setminus Z_k$ . Then one has  $t(q''_j) = f(q_j)$ ,  $t(p''_k) = p_k$ ,  $t(s''_i) = t'(s_i) = 0$ , and  $t(p''_{k,z}) = p_{k,z}$ ,  $z \in Z_k$ . Hence the composition  $ts : \mathfrak{S} \rightarrow \mathfrak{P}$  vanishes, so we have the induced  $\mathfrak{R}$ -contramodule morphism  $v : \mathfrak{V} \rightarrow \mathfrak{P}$ , where  $\mathfrak{V} = \mathfrak{T}/s(\mathfrak{S})$ .

By construction, the relations

$$\begin{aligned} \sum_{j=1}^n a_j \otimes q''_j &= \sum_{k=1}^m b_k \otimes p''_k - \sum_{k=1}^m \sum_{z \in Z_k} b_{k,z} \otimes p''_{k,z} \\ &= \sum_{k=1}^m b_k \otimes \pi_{\mathfrak{P}} \left( \sum_{x \in X_k} r_{k,x} p''_{k,x} \right) - \sum_{k=1}^m \sum_{x \in X_k} b_k r_{k,x} \otimes p''_{k,x} \end{aligned}$$

hold in  $N \otimes_{\mathbb{Z}} \mathfrak{V}$ . Hence the image of the element  $a_1 \otimes q''_1 + \dots + a_n \otimes q''_n \in N \otimes_{\mathbb{Z}} \mathfrak{V}$  vanishes in  $N \otimes_{\mathfrak{R}} \mathfrak{V}$ .

Let  $\mathfrak{U}$  denote the free left  $\mathfrak{R}$ -contramodule generated by the symbols  $u_j$ ,  $1 \leq j \leq n$ . Denote by  $u : \mathfrak{U} \rightarrow \mathfrak{Q}$  the  $\mathfrak{R}$ -contramodule morphism taking  $u_j$  to  $q_j$ , by  $q : \mathfrak{U} \rightarrow \mathfrak{T}$  the  $\mathfrak{R}$ -contramodule morphism taking  $u_j$  to  $q''_j$ , and by  $g : \mathfrak{U} \rightarrow \mathfrak{V}$  the composition

$\mathfrak{U} \rightarrow \mathfrak{T} \rightarrow \mathfrak{V}$ . We have constructed a commutative diagram

$$\begin{array}{ccc} \mathfrak{Q} & \xrightarrow{f} & \mathfrak{P} \\ u \uparrow & & \uparrow v \\ \mathfrak{U} & \xrightarrow{g} & \mathfrak{V} \end{array}$$

in the category of left  $\mathfrak{R}$ -contramodules. The objects  $\mathfrak{U}, \mathfrak{V} \in \mathfrak{R}\text{-Contra}$  are  $\lambda$ -presentable as cokernels of morphisms between coproducts of less than  $\lambda$  copies of the  $\lambda$ -presentable projective generator  $\mathfrak{R}$ . By assumption, the morphism  $u$  factorizes through  $g$ , and it follows that the image of the element  $a_1 \otimes q_1 + \cdots + a_n \otimes q_n \in N \otimes_{\mathbb{Z}} \mathfrak{Q}$  vanishes in  $N \odot_{\mathfrak{R}} \mathfrak{Q}$ .  $\square$

A left  $\mathfrak{R}$ -contramodule  $\mathfrak{F}$  is called (*contra*)*flat* if the functor of contratensor product  $N \mapsto N \odot_{\mathfrak{R}} \mathfrak{F}$  is exact on the abelian category of discrete right  $\mathfrak{R}$ -modules  $N$ .

**Lemma 5.6.** *The class of flat  $\mathfrak{R}$ -contramodules is closed under filtered colimits in  $\mathfrak{R}\text{-Contra}$ .*

*Proof.* Let  $(\mathfrak{F}_i)_{i \in \mathcal{C}}$  be a filtered diagram of flat left  $\mathfrak{R}$ -contramodules. For any discrete right  $\mathfrak{R}$ -module  $N$ , the functor of contratensor product  $N \odot_{\mathfrak{R}} - : \mathfrak{R}\text{-Contra} \rightarrow \mathbf{Ab}$  is a left adjoint, so it preserves colimits. Hence we have  $N \odot_{\mathfrak{R}} \text{colim}_i \mathfrak{F}_i = \text{colim}_i (N \odot_{\mathfrak{R}} \mathfrak{F}_i)$ . Now for any short exact sequence  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  in  $\text{Discr-}\mathfrak{R}$  and every  $i \in \mathcal{C}$  the short sequence of contratensor products  $0 \rightarrow L \odot_{\mathfrak{R}} \mathfrak{F}_i \rightarrow M \odot_{\mathfrak{R}} \mathfrak{F}_i \rightarrow N \odot_{\mathfrak{R}} \mathfrak{F}_i \rightarrow 0$  is exact in  $\mathbf{Ab}$ , since the  $\mathfrak{R}$ -contramodule  $\mathfrak{F}_i$  is flat. It remains to notice that the colimit of a filtered diagram of short exact sequences of abelian groups is a short exact sequence in order to conclude that  $\text{colim}_i \mathfrak{F}_i$  is a flat  $\mathfrak{R}$ -contramodule.  $\square$

**Examples 5.7.** (1) Let  $R$  be an arbitrary associative ring endowed with the discrete topology. Then any right  $\mathfrak{R}$ -module is discrete, so  $\text{Discr-}\mathfrak{R} = \text{Mod-}\mathfrak{R}$ , and a left  $\mathfrak{R}$ -contramodule is the same thing as a left  $\mathfrak{R}$ -module, so  $\mathfrak{R}\text{-Contra} = \mathfrak{R}\text{-Mod}$ . The contratensor product functor  $\odot_{\mathfrak{R}}$  is isomorphic to the tensor product functor  $\otimes_{\mathfrak{R}}$ , hence the flat left  $\mathfrak{R}$ -contramodules are simply the flat  $\mathfrak{R}$ -modules.

(2) Let  $\mathfrak{R}$  be a complete and separated topological associative ring where open two-sided ideals form a base of neighborhoods of zero. For any open two-sided ideal  $\mathfrak{I} \subseteq \mathfrak{R}$  and left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ , the quotient group  $\mathfrak{P}/(\mathfrak{I} \times \mathfrak{P})$  has a natural left  $\mathfrak{R}/\mathfrak{I}$ -module structure. The reduction functor  $\mathfrak{P} \mapsto \mathfrak{P}/(\mathfrak{I} \times \mathfrak{P})$  is left adjoint to the fully faithful inclusion  $\mathfrak{R}/\mathfrak{I}\text{-Mod} \rightarrow \mathfrak{R}\text{-Contra}$ . For any discrete right  $\mathfrak{R}$ -module  $N$ , the subgroup  $N_{\mathfrak{I}} \subseteq N$  of all elements annihilated by  $\mathfrak{I}$  in  $N$  is a right  $\mathfrak{R}/\mathfrak{I}$ -module. The functor  $N \mapsto N_{\mathfrak{I}}$  is right adjoint to the fully faithful inclusion  $\text{Mod-}\mathfrak{R}/\mathfrak{I} \rightarrow \text{Discr-}\mathfrak{R}$ .

The contratensor product  $N \odot_{\mathfrak{R}} \mathfrak{P}$  can be computed as the filtered colimit of tensor products over the discrete quotient rings

$$N \odot_{\mathfrak{R}} \mathfrak{P} = \text{colim}_{\mathfrak{I}} N_{\mathfrak{I}} \otimes_{\mathfrak{R}/\mathfrak{I}} (\mathfrak{P}/\mathfrak{I} \times \mathfrak{P}).$$

A left  $\mathfrak{R}$ -contramodule  $\mathfrak{F}$  is flat if and only if its reduction  $\mathfrak{F}/(\mathfrak{I} \ltimes \mathfrak{F})$  is a flat  $\mathfrak{R}/\mathfrak{I}$ -module for every open two-sided ideal  $\mathfrak{I} \subseteq \mathfrak{R}$ .

(3) Let  $\mathfrak{R} = \lim_m R/I^m$  be the  $I$ -adic completion of a Noetherian commutative ring  $R$ , as in Example 5.2. Then  $\text{Discr-}\mathfrak{R}$  is the category of  $I$ -torsion  $R$ -modules, i. e., the full subcategory in  $\text{Mod-}R$  formed by all the  $R$ -modules  $N$  such that for any  $b \in N$  and  $r \in I$  there exists an integer  $m \geq 1$  for which  $br^m = 0$ . Furthermore, one has  $\mathfrak{I} \ltimes \mathfrak{P} = \mathfrak{I}\mathfrak{P}$  for any  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$  and open ideal  $\mathfrak{I} \subseteq \mathfrak{R}$ , so  $N \odot_{\mathfrak{R}} \mathfrak{P} = N \otimes_{\mathfrak{R}} \mathfrak{P}$  for any discrete  $\mathfrak{R}$ -module  $N$  and  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ . According to [28, Lemma B.9.2], an  $\mathfrak{R}$ -contramodule is flat if and only if its underlying  $R$ -module is flat. (See [29, Section C.5] for a generalization to adic completions of noncommutative Noetherian rings by centrally generated ideals.)

An  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$  is projective if and only if  $\mathfrak{P}/I^m\mathfrak{P}$  is a projective  $R/I^m$ -module for every  $m \geq 1$  [28, Corollary B.8.2]. In particular, when  $I$  is a maximal ideal in  $R$ , an  $\mathfrak{R}$ -contramodule is flat if and only if it is projective (and if and only if it is free [28, Lemma 1.3.2]).

(4) When  $R = k[x_1, \dots, x_n]$  is the ring of polynomials in several variables over a field  $k$  and  $I$  is the ideal generated by  $x_1, \dots, x_n$  in  $R$ , the category of  $I$ -torsion  $R$ -modules  $\text{Discr-}\mathfrak{R}$  is equivalent to the category of comodules over the coalgebra  $C$  over  $k$  whose dual topological algebra  $C^* = \mathfrak{R}$  is the algebra of formal power series  $\mathfrak{R} = k[[x_1, \dots, x_n]]$ . The category of  $I$ -contramodule  $R$ -modules  $\mathfrak{R}\text{-Contra}$  is equivalent to the category of  $C$ -contramodules [27, Sections 10.1 and A.1].

More generally, for any coassociative coalgebra  $C$  over  $k$ , a right  $C$ -comodule is the same thing as a discrete right  $C^*$ -module, and a left  $C$ -contramodule is the same thing as a left  $C^*$ -contramodule [30, Sections 1.3, 1.4, and 2.3]. A  $C^*$ -contramodule is (contra)flat if and only if it is projective [27, Lemma A.3].

## 6. FLAT CONTRAMODULES

Let  $\kappa$  be a regular cardinal and  $\mathcal{N}$  a  $\kappa$ -Grothendieck abelian category. We fix the cardinal  $\kappa$  and denote by  $\text{Fun}(\mathcal{N}) \subseteq \mathbf{Ab}^{\mathcal{N}}$  the category of all additive covariant functors  $F : \mathcal{N} \rightarrow \mathbf{Ab}$  preserving  $\kappa$ -filtered colimits. Since such functors are determined by their restrictions to the full subcategory of  $\lambda$ -presentable objects in  $\mathcal{N}$  for a large enough cardinal  $\lambda$ , the category  $\text{Fun}(\mathcal{N})$  has small hom-sets (i. e., is well-defined as a category in the conventional sense of the word).

The category  $\text{Fun}(\mathcal{N})$  is abelian and cocomplete. A short sequence of functors  $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$  is exact in  $\text{Fun}(\mathcal{N})$  if and only if the short sequence of abelian groups  $0 \rightarrow E(N) \rightarrow F(N) \rightarrow G(N) \rightarrow 0$  is exact in  $\mathbf{Ab}$  for every object  $N$  in  $\mathcal{N}$ . The functors of filtered colimit are exact in  $\text{Fun}(\mathcal{N})$ .

Let  $\text{Rex}(\mathcal{N}) \subseteq \text{Fun}(\mathcal{N})$  be the full subcategory of functors preserving also finite colimits, and  $\text{Ex}(\mathcal{N}) \subseteq \text{Rex}(\mathcal{N})$  the full subcategory of functors preserving both the finite colimits and finite limits. The subcategory  $\text{Rex}(\mathcal{N})$  is closed under all colimits in  $\text{Fun}(\mathcal{N})$ , while the subcategory  $\text{Ex}(\mathcal{N})$  is closed under filtered colimits.

**Lemma 6.1.** *Let  $0 \rightarrow H \rightarrow G \rightarrow F \rightarrow 0$  be a short exact sequence in  $\text{Fun}(\mathcal{N})$ . Then*

- (a)  $H \in \text{Rex}(\mathcal{N})$  whenever  $G \in \text{Rex}(\mathcal{N})$  and  $F \in \text{Ex}(\mathcal{N})$ ;
- (b)  $H \in \text{Ex}(\mathcal{N})$  whenever  $G, F \in \text{Ex}(\mathcal{N})$ ;
- (c)  $G \in \text{Rex}(\mathcal{N})$  whenever  $F, H \in \text{Rex}(\mathcal{N})$ ;
- (d)  $G \in \text{Ex}(\mathcal{N})$  whenever  $F, H \in \text{Ex}(\mathcal{N})$ .

*Proof.* All the assertions hold due to related properties of short exact sequences of short sequences of abelian groups. In particular, part (a) holds because the kernel of a surjective morphism from a short right exact sequence to a short exact sequence of abelian groups is exact; part (b) holds because the kernel of a surjective morphism of short exact sequences is exact; part (c) holds because an extension of short right exact sequences is right exact, etc. All of these can be easily established by considering the related 9-term long exact sequence of cohomology groups.  $\square$

Let  $\mathfrak{R}$  be a complete, separated topological ring with a countable base of neighborhoods of zero formed by open right ideals; so  $\mathfrak{R}\text{-Contra}$  is a locally  $\aleph_1$ -presentable abelian category and  $\text{Discr-}\mathfrak{R}$  is a Grothendieck abelian category. Take  $\mathcal{N} = \text{Discr-}\mathfrak{R}$  and set  $\kappa = \omega$ , so  $\text{Fun}(\text{Discr-}\mathfrak{R})$  denotes the category of additive covariant functors  $\text{Discr-}\mathfrak{R} \rightarrow \mathbf{Ab}$  preserving filtered colimits and  $\text{Rex}(\text{Discr-}\mathfrak{R})$  is the category of functors  $\text{Discr-}\mathfrak{R} \rightarrow \mathbf{Ab}$  preserving all colimits. The results below in this section generalize those of [29, Section D.1], where the case of a topological ring with a countable base of neighborhoods of zero formed by open two-sided ideals was considered.

Given a left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ , we will denote by  $\text{CT}(\mathfrak{P}) : \text{Discr-}\mathfrak{R} \rightarrow \mathbf{Ab}$  the functor of contratensor product  $N \mapsto N \odot_{\mathfrak{R}} \mathfrak{P}$ . Since the contravariant functor  $N \mapsto \mathbf{Ab}(\text{CT}(\mathfrak{P}), A) = \mathfrak{R}\text{-Contra}(\mathfrak{P}, \mathbf{Ab}(N, A))$  takes colimits in  $\text{Discr-}\mathfrak{R}$  to limits in  $\mathbf{Ab}$  for any abelian group  $A$ , the functor  $\text{CT}(\mathfrak{P})$  preserves colimits. We have constructed an additive covariant functor  $\text{CT} : \mathfrak{R}\text{-Contra} \rightarrow \text{Rex}(\text{Discr-}\mathfrak{R})$ .

Conversely, let  $D$  denote the partially ordered set of open right ideals in  $\mathfrak{R}$  with respect to the inclusion order. For any open right ideal  $\mathfrak{J} \in D$ , the quotient module  $Q(\mathfrak{J}) = \mathfrak{R}/\mathfrak{J}$  is a discrete right  $\mathfrak{R}$ -module. For any pair of embedded open right ideals  $\mathfrak{J}' \subseteq \mathfrak{J} \subseteq \mathfrak{R}$ , there is a natural morphism  $\mathfrak{R}/\mathfrak{J}' \rightarrow \mathfrak{R}/\mathfrak{J}$  in  $\text{Discr-}\mathfrak{R}$  forming a commutative triangle with the projection morphisms  $\mathfrak{R} \rightarrow \mathfrak{R}/\mathfrak{J}'$  and  $\mathfrak{R} \rightarrow \mathfrak{R}/\mathfrak{J}$  in  $\text{Mod-}\mathfrak{R}$ . So we have a naturally defined diagram  $Q : D \rightarrow \text{Discr-}\mathfrak{R}$ . Given any covariant functor  $F : \text{Discr-}\mathfrak{R} \rightarrow \mathbf{Ab}$ , we set  $\text{PL}(F) = \lim_D (F \circ Q)$ .

We will say that a left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$  is *separated* if the intersection of its subgroups  $\mathfrak{J} \times \mathfrak{P}$ , taken over all the open right ideals  $\mathfrak{J}$  in  $\mathfrak{R}$ , vanishes. Clearly, any subcontramodule of a separated left  $\mathfrak{R}$ -contramodule is separated. Denote the full subcategory of separated left  $\mathfrak{R}$ -contramodules by  $\mathfrak{R}\text{-Separ} \subseteq \mathfrak{R}\text{-Contra}$ .

**Lemma 6.2.** (a) *For any additive covariant functor  $F : \text{Discr-}\mathfrak{R} \rightarrow \mathbf{Ab}$ , there is a natural structure of left  $\mathfrak{R}$ -contramodule on the set  $\text{PL}(F)$ .*

- (b) *For any additive functor  $F$ , the  $\mathfrak{R}$ -contramodule  $\text{PL}(F)$  is separated.*

(c) *The functor  $\text{PL} : \text{Fun}(\text{Discr-}\mathfrak{R}) \rightarrow \mathfrak{R}\text{-Contra}$  is right adjoint to the functor  $\text{CT} : \mathfrak{R}\text{-Contra} \rightarrow \text{Fun}(\text{Discr-}\mathfrak{R})$ .*

*Proof.* Given an additive functor  $F : \text{Discr-}\mathfrak{R} \rightarrow \mathbf{Ab}$  and an open right ideal  $\mathfrak{J} \subseteq \mathfrak{R}$ , denote by  $\psi_{\mathfrak{J}}$  the projection map  $\text{PL}(F) \rightarrow F(\mathfrak{R}/\mathfrak{J})$ . Given an element  $r \in \mathfrak{R}$ , let  $\mathfrak{J}_r \subseteq \mathfrak{R}$  be an open right ideal for which  $r\mathfrak{J}_r \subseteq \mathfrak{J}$ ; then the element  $r$  acts by a natural morphism  $r' : \mathfrak{R}/\mathfrak{J}_r \rightarrow \mathfrak{R}/\mathfrak{J}$  in the category  $\text{Discr-}\mathfrak{R}$ .

Part (a): for any set  $X$ , an  $X$ -indexed family of elements  $r_x \in \mathfrak{R}$  converging to zero in the topology of  $\mathfrak{R}$ , and an  $X$ -indexed family of elements  $p_x \in \text{PL}(F)$ , one defines the element  $\sum_x r_x p_x \in \text{PL}(F)$  by the rule

$$\psi_{\mathfrak{J}}\left(\sum_x r_x p_x\right) = \sum_x F(r'_x)(\psi_{\mathfrak{J}_x}(p_x)),$$

where  $\psi_{\mathfrak{J}_x}(p_x) \in F(\mathfrak{R}/\mathfrak{J}_{r_x})$  and  $F(r'_x) : F(\mathfrak{R}/\mathfrak{J}_{r_x}) \rightarrow F(\mathfrak{R}/\mathfrak{J})$ . The sum in the right-hand side is finite, because the morphism  $r'_x$  vanishes whenever  $r_x \in \mathfrak{J}$ . Part (b): one has  $\psi_{\mathfrak{J}}(p) = 0$  for any  $p \in \mathfrak{J} \times \text{PL}(F)$ , so  $p \in \bigcap_{\mathfrak{J}} (\mathfrak{J} \times \text{PL}(F))$  implies  $p = 0$ .

Part (c): let  $\mathfrak{P}$  be a left  $\mathfrak{R}$ -contramodule. Given a morphism  $f : \text{CT}(\mathfrak{P}) \rightarrow F$  in the category of functors  $\text{Fun}(\text{Discr-}\mathfrak{R})$ , consider the morphism of abelian groups  $f(\mathfrak{R}/\mathfrak{J}) : \mathfrak{P}/(\mathfrak{J} \times \mathfrak{P}) = \mathfrak{R}/\mathfrak{J} \odot_{\mathfrak{R}} \mathfrak{P} \rightarrow F(\mathfrak{R}/\mathfrak{J})$ . The compositions  $\mathfrak{P} \rightarrow \mathfrak{P}/(\mathfrak{J} \times \mathfrak{P}) \rightarrow F(\mathfrak{R}/\mathfrak{J})$  form a compatible cone, defining a morphism to the projective limit  $g : \mathfrak{P} \rightarrow \text{PL}(F)$ . Conversely, let  $g : \mathfrak{P} \rightarrow \text{PL}(F)$  be an  $\mathfrak{R}$ -contramodule morphism. Let  $N$  be a discrete right  $\mathfrak{R}$ -module,  $b$  an element in  $N$ , and  $\mathfrak{J}$  an open right ideal annihilating  $b$ ; then we have a related morphism  $b' : \mathfrak{R}/\mathfrak{J} \rightarrow N$  in the category  $\text{Discr-}\mathfrak{R}$ . Composing the morphism of abelian groups  $\psi_{\mathfrak{J}}g : \mathfrak{P} \rightarrow F(\mathfrak{R}/\mathfrak{J})$  with the morphism  $F(b') : F(\mathfrak{R}/\mathfrak{J}) \rightarrow F(N)$ , we obtain a morphism of abelian groups  $f'(b) : \mathfrak{P} \rightarrow F(N)$ . The desired morphism  $f(N) : N \odot_{\mathfrak{R}} \mathfrak{P} \rightarrow F(N)$  takes an element  $b \odot p \in N \odot_{\mathfrak{R}} \mathfrak{P}$  to the element  $f'(b)(p) \in F(N)$  for any  $p \in \mathfrak{P}$ .  $\square$

**Lemma 6.3.** (a) *For any functor  $F \in \text{Rex}(\text{Discr-}\mathfrak{R})$ , the adjunction morphism  $\text{CT}(\text{PL}(F)) \rightarrow F$  is an isomorphism.*

(b) *For any left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ , the adjunction morphism  $\mathfrak{P} \rightarrow \text{PL}(\text{CT}(\mathfrak{P}))$  is surjective.*

(c) *A left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$  is separated if and only if the adjunction morphism  $\mathfrak{P} \rightarrow \text{PL}(\text{CT}(\mathfrak{P}))$  is an isomorphism.*

*Proof.* Part (a): as a colimit-preserving additive functor  $\text{Discr-}\mathfrak{R} \rightarrow \mathbf{Ab}$  is determined by its restriction to the full subcategory formed by the discrete right  $\mathfrak{R}$ -modules  $\mathfrak{R}/\mathfrak{J}$ , it suffices to check that the natural morphism  $\text{PL}(F)/(\mathfrak{J} \times \text{PL}(F)) \rightarrow F(\mathfrak{R}/\mathfrak{J})$  is an isomorphism. Since the diagram  $Q$  has a countable cofinal subdiagram and the map  $F(\mathfrak{R}/\mathfrak{J}') \rightarrow F(\mathfrak{R}/\mathfrak{J})$  is surjective for any open right ideals  $\mathfrak{J}' \subseteq \mathfrak{J}$  in  $\mathfrak{R}$ , the projection map  $\psi_{\mathfrak{J}} : \text{PL}(F) \rightarrow F(\mathfrak{R}/\mathfrak{J})$  is surjective.

To compute its kernel, choose a chain of open right ideals  $\mathfrak{J} = \mathfrak{J}_0 \supseteq \mathfrak{J}_1 \supseteq \mathfrak{J}_2 \supseteq \dots$  indexed by the nonnegative integers and cofinal in  $D$ . Let  $p_n \in \text{PL}(F)$ ,  $n \geq 0$  be an

element for which  $\psi_{\mathfrak{I}_n}(p_n) = 0$ . We have an exact sequence

$$\bigoplus_{r \in \mathfrak{I}_n} \mathfrak{R}/\mathfrak{I}_r \xrightarrow{r'} \mathfrak{R}/\mathfrak{I}_{n+1} \longrightarrow \mathfrak{R}/\mathfrak{I}_n \longrightarrow 0$$

in the category  $\text{Discr-}\mathfrak{R}$ , where  $\mathfrak{I}_r$  is an open right ideal in  $\mathfrak{R}$  for which  $r\mathfrak{I}_r \subseteq \mathfrak{I}_{n+1}$ . Applying the functor  $F$ , we obtain an exact sequence of abelian groups

$$\bigoplus_{r \in \mathfrak{I}_n} F(\mathfrak{R}/\mathfrak{I}_r) \xrightarrow{F(r')} F(\mathfrak{R}/\mathfrak{I}_{n+1}) \longrightarrow F(\mathfrak{R}/\mathfrak{I}_n) \longrightarrow 0.$$

Hence there exists finite families of elements  $r_{n,1}, \dots, r_{n,k_n} \in \mathfrak{I}_{n+1}$  and  $q'_{n,i} \in F(\mathfrak{R}/\mathfrak{I}_{r_{n,i}})$  such that  $\psi_{\mathfrak{I}_{n+1}}(p_n) = F(r'_{n,1})(q'_{n,1}) + \dots + F(r'_{n,k_n})(q'_{n,k_n})$  in  $F(\mathfrak{R}/\mathfrak{I}_{n+1})$ . Lifting the elements  $q'_{n,i}$  to  $q_{n,i} \in \text{PL}(F)$  with  $\psi_{\mathfrak{I}_{r_{n,i}}}(q_{n,i}) = q'_{n,i}$ , we set  $p_{n+1} = p_n - r_{n,1}q_{n,1} - \dots - r_{n,k_n}q_{n,k_n}$ , so  $\psi_{\mathfrak{I}_{n+1}}(p_{n+1}) = 0$ . Starting from an element  $p_0 = p \in \text{PL}(F)$  such that  $\psi_{\mathfrak{I}}(p) = 0$  and proceeding by induction in this way, we produce a family of elements  $r_{n,i} \in \mathfrak{I}$  converging to zero in the topology of  $\mathfrak{R}$  and a family of elements  $q_{n,i} \in \text{PL}(F)$  for which  $p = \pi_{\text{PL}(F)}(\sum_{n,i} r_{n,i}q_{n,i})$ . Hence  $p \in \mathfrak{I} \ltimes \text{PL}(F)$ .

Parts (b–c): by the definition,  $\text{PL}(\text{CT}(\mathfrak{P})) = \lim_{\mathfrak{I}} \mathfrak{P}/(\mathfrak{I} \ltimes \mathfrak{P})$ , so we only have to prove that the natural map to the projective limit of quotients  $\mathfrak{P} \rightarrow \lim_{\mathfrak{I}} \mathfrak{P}/(\mathfrak{I} \ltimes \mathfrak{P})$  is surjective for any left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ . Let  $p = (p_{\mathfrak{I}})_{\mathfrak{I} \subseteq \mathfrak{R}}$  be an element of the projective limit. Lift the elements  $p_{\mathfrak{I}} \in \mathfrak{P}/(\mathfrak{I} \ltimes \mathfrak{P})$  to some elements  $p'_{\mathfrak{I}} \in \mathfrak{P}$ . Let  $\mathfrak{I}_0 \supseteq \mathfrak{I}_1 \supseteq \mathfrak{I}_2 \supseteq \dots$  be a chain of open right ideals indexed by the nonnegative integers and cofinal in  $D$ . For any  $n \geq 0$ , the difference  $p'_{\mathfrak{I}_{n+1}} - p'_{\mathfrak{I}_n}$  belongs to  $\mathfrak{I}_n \ltimes \mathfrak{P}$ . Hence there is a set  $X_n$ , an  $X_n$ -indexed family of elements  $r_{n,x} \in \mathfrak{I}_n$  converging to zero in the topology of  $\mathfrak{R}$ , and an  $X_n$ -indexed family of elements  $q_{n,x} \in \mathfrak{P}$  such that  $p'_{\mathfrak{I}_{n+1}} - p'_{\mathfrak{I}_n} = \pi_{\mathfrak{P}}(\sum_{x \in X_n} r_{n,x}q_{n,x})$ . Let  $X = \coprod_n X_n$  be the disjoint union of the sets  $X_n$ ; then the  $X$ -indexed family of elements  $r_{n,x} \in \mathfrak{R}$  converges to zero in the topology of  $\mathfrak{R}$ . Set  $q = p'_{\mathfrak{I}_0} + \pi_{\mathfrak{P}}(\sum_{(n,x) \in X} r_{n,x}q_{n,x}) \in \mathfrak{P}$ . Then the difference  $q - p'_{\mathfrak{I}_n}$  belongs to  $\mathfrak{I}_n \ltimes \mathfrak{P}$  for every  $n \geq 0$ , so the element  $q$  is a preimage of the projective limit element  $p$  in the contramodule  $\mathfrak{P}$ .  $\square$

**Corollary 6.4.** *The restrictions of the functors PL and CT are mutually inverse equivalences between the full subcategory of colimit-preserving functors  $\text{Rex}(\text{Discr-}\mathfrak{R}) \subseteq \text{Fun}(\text{Discr-}\mathfrak{R})$  and the full subcategory of separated contramodules  $\mathfrak{R}\text{-Separ} \subseteq \mathfrak{R}\text{-Contra}$ .*  $\square$

**Remark 6.5.** Given a small category  $\mathcal{A}$  and a regular cardinal  $\lambda$ , denote by  $\text{Lex}_{\lambda}(\mathcal{A})$  the category of all covariant functors  $\mathcal{A} \rightarrow \mathbf{Set}$  preserving all  $\lambda$ -small limits existing in  $\mathcal{A}$ . The category  $\text{Lex}_{\lambda}(\mathcal{A})$  has all kinds of good properties; in particular, it is a locally  $\lambda$ -presentable category and a reflective subcategory of the category  $\mathbf{Set}^{\mathcal{A}}$  of all covariant functors  $\mathcal{A} \rightarrow \mathbf{Set}$ . Moreover, any locally  $\lambda$ -presentable category  $\mathcal{K}$  is equivalent to  $\text{Lex}_{\lambda}(\mathcal{K}_{\lambda}^{\text{op}})$ , where  $\mathcal{K}_{\lambda}$  is the full subcategory of  $\lambda$ -presentable objects in  $\mathcal{K}$  [1, Theorem 1.46]. The construction of the reflector  $\mathbf{Set}^{\mathcal{A}} \rightarrow \text{Lex}_{\lambda}(\mathcal{A})$  involves directed colimits over chains of length  $\lambda$  in the category  $\mathbf{Set}$  [1, Construction 1.37].

Furthermore, suppose that  $\mathcal{A}$  is an additive category. Then any functor  $F : \mathcal{A} \rightarrow \mathbf{Set}$  preserving finite products lifts to an additive functor  $F' : \mathcal{A} \rightarrow \mathbf{Ab}$  in a unique way. Limits in  $\mathbf{Ab}$  agree with those in  $\mathbf{Set}$ , so the category  $\mathrm{Lex}_\lambda(\mathcal{A})$  can be equivalently defined as the category of all covariant additive functors  $\mathcal{A} \rightarrow \mathbf{Ab}$  preserving  $\lambda$ -small limits. When the category  $\mathcal{A}$  is abelian, the category  $\mathrm{Lex}_\omega(\mathcal{A})$  of all left exact functors  $\mathcal{A} \rightarrow \mathbf{Ab}$  is abelian, too, and is a localization of the abelian category  $\mathrm{Fadd}(\mathcal{A})$  of all additive functors  $\mathcal{A} \rightarrow \mathbf{Ab}$ . So the inclusion  $\mathrm{Lex}_\omega(\mathcal{A}) \rightarrow \mathrm{Fadd}(\mathcal{A})$  is a left exact functor, and the functor  $\mathrm{Fadd}(\mathcal{A}) \rightarrow \mathrm{Lex}_\omega(\mathcal{A})$  left adjoint to the inclusion is exact. The construction of the latter functor involves filtered colimits in the category  $\mathbf{Ab}$ . This can be viewed as a kind of “additive sheaf theory”:  $\mathrm{Lex}_\omega(\mathcal{A})$  is interpreted as the category of additive sheaves on the category  $\mathcal{A}^{\mathrm{op}}$  endowed with the Grothendieck topology with epimorphisms in  $\mathcal{A}^{\mathrm{op}}$  playing the role of generating coverings.  $\mathrm{Fadd}(\mathcal{A})$  is the category of additive presheaves, and  $\mathrm{Fadd}(\mathcal{A}) \rightarrow \mathrm{Lex}_\omega(\mathcal{A})$  is the sheafification functor [11, Appendix A].

Unlike the limits, the colimits are not preserved by the forgetful functor  $\mathbf{Ab} \rightarrow \mathbf{Set}$ , so colimit-preserving functors  $\mathcal{A} \rightarrow \mathbf{Set}$  are entirely unrelated to colimit-preserving functors  $\mathcal{A} \rightarrow \mathbf{Ab}$ . Still, for an additive category  $\mathcal{A}$ , any functor  $\mathcal{A} \rightarrow \mathbf{Ab}$  preserving finite coproducts is additive. For any small category  $\mathcal{A}$  one can consider the category  $\mathrm{Rex}_\lambda(\mathcal{A})$  of covariant functors  $\mathcal{A} \rightarrow \mathbf{Ab}$  preserving  $\lambda$ -small colimits. However, the category  $\mathrm{Rex}_\omega(\mathcal{A})$  does not have good properties. To be sure, let  $R$  be an associative ring and  $\mathcal{N} = \mathrm{Mod}\text{-}R$  the category of right  $R$ -modules. Then the category  $\mathrm{Rex}(\mathcal{N})$  of colimit-preserving functors  $\mathcal{N} \rightarrow \mathbf{Ab}$  is equivalent to the category of left  $R$ -modules  $R\text{-Mod}$ . Furthermore, if  $\mathcal{A} \subset \mathcal{N}$  is the category of finitely presented right  $R$ -modules, then one has  $\mathrm{Rex}_\omega(\mathcal{A}) = \mathrm{Rex}(\mathcal{N}) = R\text{-Mod}$ , so the category  $\mathrm{Rex}_\omega(\mathcal{A})$  is abelian. But in general it is not, even for an abelian category  $\mathcal{A}$ . The explanation is that  $\mathrm{Rex}_\omega(\mathcal{A})$  is a kind of “additive cosheaf category”, and cosheaves are known to be problematic. In particular, an attempt to dualize the sheaf theory would presume constructing cosheafification in terms of filtered limits in  $\mathbf{Ab}$ , and these are not exact functors (see the discussion in [14, Section 2.5] and the introduction to [29]).

Specifically, let  $R$  be a Noetherian commutative ring,  $I = (r)$  a principal ideal, and  $\mathfrak{R} = \lim_m R/I^m$  the adic completion. Let  $\mathcal{N} = \mathrm{Discr}\text{-}\mathfrak{R}$  be the category of  $(r)$ -torsion  $R$ -modules (see Example 5.7(3)) and  $\mathcal{A} \subset \mathcal{N}$  the full subcategory of finitely generated  $(r)$ -torsion  $R$ -modules. Then  $\mathcal{N}$  is a locally Noetherian abelian category and  $\mathcal{A}$  is a Noetherian abelian category, but the category  $\mathrm{Rex}_\omega(\mathcal{A}) = \mathrm{Rex}(\mathcal{N})$  is not abelian. Indeed, by Corollary 6.4,  $\mathrm{Rex}(\mathcal{N})$  is equivalent to the category of separated  $\mathfrak{R}$ -contramodules  $\mathfrak{R}\text{-Separ}$ . A well-known counterexample demonstrates that the category  $\mathfrak{R}\text{-Separ}$  is not well-behaved. As a full subcategory in  $\mathfrak{R}\text{-Contra}$ , it is closed neither under the cokernels of monomorphisms nor under extensions (see [38, Example 2.5] or [30, Section 1.5]). All limits and colimits exist in  $\mathfrak{R}\text{-Separ}$ , but the same counterexample shows that the kernel of cokernel may differ from the cokernel of kernel, so  $\mathfrak{R}\text{-Separ}$  is not abelian. The abelian category  $\mathfrak{R}\text{-Contra}$  may be viewed as the “corrected” or “right” version of the cosheaf category  $\mathrm{Rex}(\mathcal{N})$ .



Still, we have to consider colimit-preserving functors  $\text{Discr-}\mathfrak{R} \rightarrow \mathbf{Ab}$  and separated contramodules in this section for technical purposes.

**Lemma 6.6.** *Let  $0 \rightarrow H \rightarrow G \rightarrow F \rightarrow 0$  be a short exact sequence in  $\text{Fun}(\text{Discr-}\mathfrak{R})$ . Assume that  $H \in \text{Rex}(\text{Discr-}\mathfrak{R})$ . Then the short sequence  $0 \rightarrow \text{PL}(H) \rightarrow \text{PL}(G) \rightarrow \text{PL}(F) \rightarrow 0$  is exact in  $\mathfrak{R}\text{-Contra}$ .*

*Proof.* For any open right ideal  $\mathfrak{J}$  in  $\mathfrak{R}$ , we have a short exact sequence of abelian groups  $0 \rightarrow H(\mathfrak{R}/\mathfrak{J}) \rightarrow G(\mathfrak{R}/\mathfrak{J}) \rightarrow F(\mathfrak{R}/\mathfrak{J}) \rightarrow 0$ . For any pair of embedded open right ideals  $\mathfrak{J}' \subseteq \mathfrak{J}$ , the morphism  $\mathfrak{R}/\mathfrak{J}' \rightarrow \mathfrak{R}/\mathfrak{J}$  is an epimorphism in  $\text{Discr-}\mathfrak{R}$ , so the induced morphism  $H(\mathfrak{R}/\mathfrak{J}') \rightarrow H(\mathfrak{R}/\mathfrak{J})$  is an epimorphism in  $\mathbf{Ab}$ . Since the diagram  $Q$  has a countable cofinal subdiagram, it follows that the derived functor  $\lim_{\mathfrak{J}}^1 H(\mathfrak{R}/\mathfrak{J})$  vanishes. Passing to the limit, we obtain a short exact sequence of abelian groups  $0 \rightarrow \lim_{\mathfrak{J}} H(\mathfrak{R}/\mathfrak{J}) \rightarrow \lim_{\mathfrak{J}} G(\mathfrak{R}/\mathfrak{J}) \rightarrow \lim_{\mathfrak{J}} F(\mathfrak{R}/\mathfrak{J}) \rightarrow 0$ , which means the desired short exact sequence of left  $\mathfrak{R}$ -contramodules.  $\square$

By the definition, a left  $\mathfrak{R}$ -contramodule  $\mathfrak{F}$  is flat if and only if the functor  $\text{CT}(\mathfrak{F}) \in \text{Rex}(\text{Discr-}\mathfrak{R})$  belongs to the full subcategory  $\text{Ex}(\text{Discr-}\mathfrak{R}) \subseteq \text{Rex}(\text{Discr-}\mathfrak{R})$ . We will see below in Corollary 6.15 that any flat  $\mathfrak{R}$ -contramodule is separated.

**Lemma 6.7.** *Let  $0 \rightarrow \mathfrak{Q} \rightarrow \mathfrak{P} \rightarrow \mathfrak{F} \rightarrow 0$  be a short exact sequence of left  $\mathfrak{R}$ -contramodules. Assume that the contramodule  $\mathfrak{P}$  is separated, while the contramodule  $\mathfrak{F}$  is flat and separated. Then for any discrete right  $\mathfrak{R}$ -module  $N$  the short sequence of abelian groups  $0 \rightarrow N \odot_{\mathfrak{R}} \mathfrak{Q} \rightarrow N \odot_{\mathfrak{R}} \mathfrak{P} \rightarrow N \odot_{\mathfrak{R}} \mathfrak{F} \rightarrow 0$  is exact.*

*Proof.* Set  $H = \text{CT}(\mathfrak{Q})$ ,  $G = \text{CT}(\mathfrak{P})$ , and  $F = \text{CT}(\mathfrak{F})$ . The functor of contratensor product  $N \odot_{\mathfrak{R}} - : \mathfrak{R}\text{-Contra} \rightarrow \mathbf{Ab}$  is a left adjoint, so it preserves colimits. Hence the sequence of functors  $H \rightarrow G \rightarrow F \rightarrow 0$  is exact in  $\text{Fun}(\text{Discr-}\mathfrak{R})$ . Denote by  $H'$  the kernel of the morphism  $G \rightarrow F$  in  $\text{Fun}(\text{Discr-}\mathfrak{R})$ ; then the morphism  $H \rightarrow G$  decomposes as  $H \rightarrow H' \rightarrow G$ . The functor  $F$  belongs to the subcategory  $\text{Ex}(\text{Discr-}\mathfrak{R}) \subseteq \text{Fun}(\text{Discr-}\mathfrak{R})$  and the functor  $G$  belongs to  $\text{Rex}(\text{Discr-}\mathfrak{R})$ , so by Lemma 6.1(a) the functor  $H'$  belongs to  $\text{Rex}(\text{Discr-}\mathfrak{R})$ .

Following Lemma 6.6, we have a short exact sequence of left  $\mathfrak{R}$ -contramodules  $0 \rightarrow \text{PL}(H') \rightarrow \text{PL}(G) \rightarrow \text{PL}(F) \rightarrow 0$ . The  $\mathfrak{R}$ -contramodule  $\mathfrak{Q}$  is separated as a subcontramodule of a separated  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ , so all the three adjunction morphisms  $\mathfrak{Q} \rightarrow \text{PL}(H)$ ,  $\mathfrak{P} \rightarrow \text{PL}(G)$ , and  $\mathfrak{F} \rightarrow \text{PL}(F)$  are isomorphisms of  $\mathfrak{R}$ -contramodules by Lemma 6.3(c). We can conclude that the morphism  $\text{PL}(H) \rightarrow \text{PL}(H')$  is an isomorphism of  $\mathfrak{R}$ -contramodules. Since both  $H$  and  $H'$  belong to  $\text{Rex}(\text{Discr-}\mathfrak{R})$ , by Corollary 6.4 it follows that the morphism of functors  $H \rightarrow H'$  is an isomorphism and the short sequence  $0 \rightarrow H \rightarrow G \rightarrow F \rightarrow 0$  is exact.  $\square$

**Corollary 6.8.** *The kernel of any surjective morphism of flat and separated left  $\mathfrak{R}$ -contramodules is flat and separated.*

*Proof.* Follows from Lemmas 6.7 and 6.1(b).  $\square$

**Lemma 6.9.** *Any projective left  $\mathfrak{R}$ -contramodule is flat and separated.*

*Proof.* It suffices to show that free left  $\mathfrak{R}$ -contramodules  $\mathfrak{R}[[X]]$  are flat and separated. Indeed, one has  $N \odot_{\mathfrak{R}} \mathfrak{R} = N$  for any discrete right  $\mathfrak{R}$ -module  $N$ , so  $\mathrm{CT}(\mathfrak{R})$  is simply the forgetful functor  $\mathrm{Discr}\text{-}\mathfrak{R} \rightarrow \mathbf{Ab}$ . The functor  $\mathrm{CT}$  is a left adjoint, so it preserves coproducts, hence  $N \odot_{\mathfrak{R}} \mathfrak{R}[[X]] = N^{(X)}$  is the coproduct of  $X$  copies of the underlying abelian group of a discrete right  $\mathfrak{R}$ -module  $N$ . This functor preserves all colimits and finite limits, thus the left  $\mathfrak{R}$ -contramodule  $\mathfrak{R}[[X]]$  is flat. To prove that it is separated, one computes that  $\mathfrak{I} \times \mathfrak{R}[[X]] = \mathfrak{I}[[X]] \subseteq \mathfrak{R}[[X]]$  for any open right ideal  $\mathfrak{I}$  in  $\mathfrak{R}$ , and  $\bigcap_{\mathfrak{I}} \mathfrak{I}[[X]] = 0$  because  $\bigcap_{\mathfrak{I}} \mathfrak{I} = 0$  in  $\mathfrak{R}$  (since  $\mathfrak{R}$  is a separated topological ring).  $\square$

The next lemma only differs from Lemma 6.7 in that the separatedness condition on the middle term of the exact sequence of contramodules is dropped.

**Lemma 6.10.** *Let  $0 \rightarrow \mathfrak{Q} \rightarrow \mathfrak{P} \rightarrow \mathfrak{F} \rightarrow 0$  be a short exact sequence of left  $\mathfrak{R}$ -contramodules. Assume the  $\mathfrak{R}$ -contramodule  $\mathfrak{F}$  is flat and separated. Then the short sequence of functors  $0 \rightarrow \mathrm{CT}(\mathfrak{Q}) \rightarrow \mathrm{CT}(\mathfrak{P}) \rightarrow \mathrm{CT}(\mathfrak{F}) \rightarrow 0$  is exact in  $\mathrm{Fun}(\mathrm{Discr}\text{-}\mathfrak{R})$ .*

*Proof.* Given a discrete right  $\mathfrak{R}$ -module  $N$ , the left derived functors of contratensor product  $\mathrm{Ctrtor}_n^{\mathfrak{R}}(N, -) : \mathfrak{R}\text{-Contra} \rightarrow \mathbf{Ab}$ ,  $n \geq 0$  are defined as the homology groups

$$\mathrm{Ctrtor}_n^{\mathfrak{R}}(N, \mathfrak{Q}) = H_n(N \odot_{\mathfrak{R}} \mathfrak{P}_{\bullet})$$

of the contratensor product of  $N$  with a left resolution

$$\mathfrak{P}_{\bullet} = (\cdots \rightarrow \mathfrak{P}_2 \rightarrow \mathfrak{P}_1 \rightarrow \mathfrak{P}_0) \rightarrow \mathfrak{Q}$$

of an  $\mathfrak{R}$ -contramodule  $\mathfrak{Q}$  by projective  $\mathfrak{R}$ -contramodules  $\mathfrak{P}_n$ . Since the functor  $\mathfrak{Q} \mapsto N \odot_{\mathfrak{R}} \mathfrak{Q}$  is right exact, one has  $\mathrm{Ctrtor}_0^{\mathfrak{R}}(N, \mathfrak{Q}) = N \odot_{\mathfrak{R}} \mathfrak{Q}$ . This definition of a left derived functor provides for any short exact sequence of left  $\mathfrak{R}$ -contramodules  $0 \rightarrow \mathfrak{Q} \rightarrow \mathfrak{P} \rightarrow \mathfrak{F} \rightarrow 0$  a long exact sequence of abelian groups

$$\begin{aligned} \cdots \rightarrow \mathrm{Ctrtor}_1^{\mathfrak{R}}(N, \mathfrak{P}) \rightarrow \mathrm{Ctrtor}_1^{\mathfrak{R}}(N, \mathfrak{F}) \\ \rightarrow N \odot_{\mathfrak{R}} \mathfrak{Q} \rightarrow N \odot_{\mathfrak{R}} \mathfrak{P} \rightarrow N \odot_{\mathfrak{R}} \mathfrak{F} \rightarrow 0. \end{aligned}$$

In particular, let  $\mathfrak{F}$  be a flat and separated  $\mathfrak{R}$ -contramodule. Taking  $\mathfrak{P}$  to be a projective  $\mathfrak{R}$ -contramodule, we have  $\mathrm{Ctrtor}_1^{\mathfrak{R}}(N, \mathfrak{P}) = 0$  by the definition. By Lemmas 6.9 and 6.7, the morphism  $N \odot_{\mathfrak{R}} \mathfrak{Q} \rightarrow N \odot_{\mathfrak{R}} \mathfrak{P}$  is injective. Hence  $\mathrm{Ctrtor}_1^{\mathfrak{R}}(N, \mathfrak{F}) = 0$ , and it follows from the same long exact sequence that the short sequence  $0 \rightarrow N \odot_{\mathfrak{R}} \mathfrak{Q} \rightarrow N \odot_{\mathfrak{R}} \mathfrak{P} \rightarrow N \odot_{\mathfrak{R}} \mathfrak{F} \rightarrow 0$  is exact for an arbitrary  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ .  $\square$

**Remark 6.11.** Applying Lemma 6.7 together with Corollary 6.8 and Lemma 6.9, one shows easily that  $\mathrm{Ctrtor}_n^{\mathfrak{R}}(N, \mathfrak{F}) = 0$  for any flat and separated left  $\mathfrak{R}$ -contramodule  $\mathfrak{F}$ , any discrete right  $\mathfrak{R}$ -module  $N$ , and all  $n \geq 1$ .

**Remark 6.12.** The conventional derived functor  $\mathrm{Tor}_*^R(-, -)$  over an associative ring  $R$  can be computed by taking a flat projective resolution of either of its arguments; the derived functors obtained by resolving the first and the second argument of the tensor

product functor  $\otimes_R$  are naturally isomorphic. This important feature of the classical theory is absent in the case of the  $\text{Ctrtor}$ , as there may be no flat or projective objects in the category of discrete right  $\mathfrak{R}$ -modules, in general. So the functor  $\text{Ctrtor}_*^{\mathfrak{R}}(-, -)$  can be only computed by resolving the second (contramodule) argument of the functor of contratensor product  $\odot_{\mathfrak{R}}$ , generally speaking. This is the reason why the proofs of Lemmas 6.7 and 6.10, as well as of Remark 6.11, are so much more indirect and intricate than in the case of a discrete ring  $R$  (cf. [27, first Question in Section 5.3]).

**Corollary 6.13.** *The class of flat and separated  $\mathfrak{R}$ -contramodules is closed under extensions in the abelian category  $\mathfrak{R}\text{-Contra}$ .*

*Proof.* Let  $0 \rightarrow \mathfrak{Q} \rightarrow \mathfrak{P} \rightarrow \mathfrak{F} \rightarrow 0$  be a short exact sequence of left  $\mathfrak{R}$ -contramodules. Assume that the contramodules  $\mathfrak{Q}$  and  $\mathfrak{F}$  are flat and separated. According to Lemma 6.10, the short sequence of functors  $0 \rightarrow \text{CT}(\mathfrak{Q}) \rightarrow \text{CT}(\mathfrak{P}) \rightarrow \text{CT}(\mathfrak{F}) \rightarrow 0$  is exact in  $\text{Fun}(\text{Discr-}\mathfrak{R})$ . Applying Lemma 6.1(d), we can conclude that  $\mathfrak{P}$  is a flat  $\mathfrak{R}$ -contramodule. To prove that  $\mathfrak{P}$  is separated, consider the short sequence of  $\mathfrak{R}$ -contramodules  $0 \rightarrow \text{PL}(\text{CT}(\mathfrak{Q})) \rightarrow \text{PL}(\text{CT}(\mathfrak{P})) \rightarrow \text{PL}(\text{CT}(\mathfrak{F})) \rightarrow 0$ , which is exact by Lemma 6.6. According to Lemma 6.3(c), both the adjunction morphisms  $\mathfrak{Q} \rightarrow \text{PL}(\text{CT}(\mathfrak{Q}))$  and  $\mathfrak{F} \rightarrow \text{PL}(\text{CT}(\mathfrak{F}))$  are isomorphisms. It follows that the adjunction morphism  $\mathfrak{P} \rightarrow \text{PL}(\text{CT}(\mathfrak{P}))$  is an isomorphism, too.  $\square$

The following result is a version of Nakayama's lemma for contramodules (see [28, Lemma 1.3.1] for a formulation more similar to the classical Nakayama's lemma).

**Lemma 6.14.** *If  $\mathfrak{P}$  is a left  $\mathfrak{R}$ -contramodule, and  $\text{CT}(\mathfrak{P}) = 0$ , then  $\mathfrak{P} = 0$ .*

*Proof.* We have to show that a left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$  vanishes whenever  $\mathfrak{P} = \mathfrak{I} \ltimes \mathfrak{P}$  for every open right ideal  $\mathfrak{I}$  in  $\mathfrak{R}$ . Let  $p \in \mathfrak{P}$  be an element, and let  $\mathfrak{I}_1 \supseteq \mathfrak{I}_2 \supseteq \dots$  be a chain of open right ideals, indexed by the integers, forming a base of neighborhoods of zero in  $\mathfrak{R}$ . For every  $n \geq 1$ , choose a section  $s_n$  of the surjective map  $\pi_{\mathfrak{P}}|_{\mathfrak{I}_n[[\mathfrak{P}]]} : \mathfrak{I}_n[[\mathfrak{P}]] \rightarrow \mathfrak{P}$ ; so  $s_n : \mathfrak{P} \rightarrow \mathfrak{I}_n[[\mathfrak{P}]]$  is a map of sets for which  $\pi_{\mathfrak{P}} s_n = \text{id}_{\mathfrak{P}}$ .

Set  $n_1 = 1$  and  $p_1 = s_{n_1}(p) \in \mathfrak{I}_{n_1}[[\mathfrak{P}]]$ . Among the coefficients  $r \in \mathfrak{I}_1$  appearing in the infinite formal linear combination  $p_1$ , there is only a finite set  $R_2$  of those which do not belong to  $\mathfrak{I}_2$ . Let  $n_2 \geq 2$  be an integer such that  $r\mathfrak{I}_{n_2} \subseteq \mathfrak{I}_2$  for every  $r \in R_2$ . Put  $p_2 = \mathfrak{I}_{n_1}[[s_{n_2}]](p_1)$ . Then one has  $p_2 \in \mathfrak{I}_{n_1}[[\mathfrak{I}_{n_2}[[\mathfrak{P}]]]]$ ,  $p_1 = \mathfrak{I}_{n_1}[[\pi_{\mathfrak{P}}]](p_2)$ , and  $\phi_{\mathfrak{P}}(p_2) \in \mathfrak{I}_2[[\mathfrak{P}]]$ . Put  $q_2 = p_2$ .

Among the coefficients  $r \in \mathfrak{I}_2$  appearing in the infinite formal linear combination  $\phi_{\mathfrak{P}}(p_2)$ , there is only a finite set  $R_3$  of those which do not belong to  $\mathfrak{I}_3$ . Let  $n_3 \geq 3$  be an integer such that  $r\mathfrak{I}_{n_3} \subseteq \mathfrak{I}_3$  for every  $r \in R_3$ . Put  $p_3 = \mathfrak{I}_{n_1}[[\mathfrak{I}_{n_2}[[s_{n_3}]]]](p_2)$ . Then one has  $p_3 \in \mathfrak{I}_{n_1}[[\mathfrak{I}_{n_2}[[\mathfrak{I}_{n_3}[[\mathfrak{P}]]]]]]$ ,  $p_2 = \mathfrak{I}_{n_1}[[\mathfrak{I}_{n_2}[[\pi_{\mathfrak{P}}]]]](p_3)$ ,  $\phi_{\mathfrak{P}}|_{\mathfrak{I}_{n_3}[[\mathfrak{P}]]}(p_3) \in \mathfrak{I}_2[[\mathfrak{I}_{n_3}[[\mathfrak{P}]]]]$ , and  $\phi_{\mathfrak{P}}\phi_{\mathfrak{I}_{n_3}[[\mathfrak{P}]]}(p_3) \in \mathfrak{I}_3[[\mathfrak{P}]]$ . Put  $q_3 = \phi_{\mathfrak{I}_{n_3}[[\mathfrak{P}]]}(p_3)$ .

Among the coefficients  $r \in \mathfrak{I}_3$  appearing in the infinite formal linear combination  $\phi_{\mathfrak{P}}(q_3) \in \mathfrak{I}_3[[\mathfrak{P}]]$ , there is only a finite set  $R_4$  of those which do not belong to  $\mathfrak{I}_4$ . Let  $n_4 \geq 4$  be an integer such that  $r\mathfrak{I}_{n_4} \subseteq \mathfrak{I}_4$  for every  $r \in R_4$ . Put  $p_4 = \mathfrak{I}_{n_1}[[\mathfrak{I}_{n_2}[[\mathfrak{I}_{n_3}[[s_{n_4}]]]]]](p_3)$  and  $q_4 = \phi_{\mathfrak{I}_{n_4}[[\mathfrak{P}]]}\phi_{\mathfrak{I}_{n_3}[[\mathfrak{I}_{n_4}[[\mathfrak{P}]]]]}(p_4) \in \mathfrak{I}_3[[\mathfrak{I}_{n_4}[[\mathfrak{P}]]]]$ , etc.

For any set  $X$ , define inductively  $\mathfrak{R}^{(0)}[[X]] = X$  and  $\mathfrak{R}^{(k)}[[X]] = \mathfrak{R}^{(k-1)}[[\mathfrak{R}[[X]]]]$  for  $k \geq 1$ . Let  $\phi_X^{(k)} : \mathfrak{R}^{(k)}[[X]] \rightarrow \mathfrak{R}[[X]]$  denote the iterated monad multiplication map (opening the outer  $k - 1$  pairs of parentheses or all but the innermost one pair, i. e., all the parentheses that can be opened for an arbitrary set  $X$ ). Proceeding as above, we construct a sequence of integers  $n_k \geq k$  and a sequence of elements  $p_k \in \mathfrak{I}_{n_1}[[\cdots [[\mathfrak{I}_{n_k}[[\mathfrak{P}]]]] \cdots]]$  such that  $p_k = \mathfrak{I}_{n_1}[[\cdots [[\mathfrak{I}_{n_{k-1}}[[s_{n_k}]]]] \cdots]](p_{k-1})$  and the image of  $p_k$  under the map  $\phi_{\mathfrak{P}}^{(k)} : \mathfrak{R}^{(k)}[[\mathfrak{P}]] \rightarrow \mathfrak{R}[[\mathfrak{P}]]$  belongs to  $\mathfrak{I}_k[[\mathfrak{P}]]$ .

Then one has  $p_{k-1} = \mathfrak{I}_{n_1}[[\cdots [[\mathfrak{I}_{n_{k-1}}[[\pi_{\mathfrak{P}}]]]] \cdots]](p_k)$  and the image  $q_k = \phi_{\mathfrak{I}_{n_k}[[\mathfrak{P}]]}^{(k-1)}(p_k)$  of the element  $p_k$ ,  $k \geq 2$  under the iterated monad multiplication map

$$\phi_{\mathfrak{I}_{n_k}[[\mathfrak{P}]]} \phi_{\mathfrak{I}_{n_{k-1}}[[\mathfrak{I}_{n_k}[[\mathfrak{P}]]]]} \cdots \phi_{\mathfrak{I}_{n_3}[[\cdots [[\mathfrak{I}_{n_k}[[\mathfrak{P}]]]] \cdots]]} : \mathfrak{I}_{n_1}[[\cdots [[\mathfrak{I}_{n_k}[[\mathfrak{P}]]]] \cdots]] \longrightarrow \mathfrak{R}[[\mathfrak{I}_{n_k}[[X]]]]$$

opening the outer  $k - 2$  pairs of parentheses (i. e., all but the innermost two pairs) belongs to  $\mathfrak{I}_{k-1}[[\mathfrak{I}_{n_k}[[\mathfrak{P}]]]]$ . The elements  $q_k$  satisfy the equations

$$\phi_{\mathfrak{P}}(q_k) = \phi_{\mathfrak{P}}^{(k)}(p_k) = \mathfrak{I}_k[[\pi_{\mathfrak{P}}]](q_{k+1}), \quad k \geq 2.$$

For any set  $X$ , we endow the abelian group  $\mathfrak{R}[[X]]$  with the topology of the limit of discrete groups  $\lim_{\mathfrak{I}} \mathfrak{R}/\mathfrak{I}[X]$ . For any map of sets  $f : X \rightarrow Y$ , the induced map  $\mathfrak{R}[[f]] : \mathfrak{R}[[X]] \rightarrow \mathfrak{R}[[Y]]$  is a continuous homomorphism of topological groups. In particular, the set  $\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]]$  is endowed with the topological group structure of  $\mathfrak{R}[[X]]$ , where  $X = \mathfrak{R}[[\mathfrak{P}]]$  is viewed as an abstract set, while the set  $\mathfrak{R}[[\mathfrak{P}]]$  is endowed with the topological group structure of  $\mathfrak{R}[[Y]]$  with the set  $Y = \mathfrak{P}$ . Then both the maps  $\mathfrak{R}[[\pi_{\mathfrak{P}}]]$ ,  $\phi_{\mathfrak{P}} : \mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{P}]]$  are continuous group homomorphisms.

The sum  $\sum_{k=2}^{\infty} q_k$  converges in the topology of  $\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]]$ , since  $q_k \in \mathfrak{I}_{k-1}[[\mathfrak{R}[[\mathfrak{P}]]]]$ . Therefore, we have

$$\mathfrak{R}[[\pi_{\mathfrak{P}}]] \left( \sum_{k=2}^{\infty} q_k \right) - \phi_{\mathfrak{P}} \left( \sum_{k=2}^{\infty} q_k \right) = \mathfrak{R}[[\pi_{\mathfrak{P}}]](q_2) = p_1,$$

hence  $p = \pi_{\mathfrak{P}}(p_1) = 0$  by the contraassociativity equation.  $\square$

**Corollary 6.15.** *Any flat left  $\mathfrak{R}$ -contramodule is separated.*

*Proof.* Let  $\mathfrak{P}$  be a flat left  $\mathfrak{R}$ -contramodule. Set  $\mathfrak{F} = \text{PL}(\text{CT}(\mathfrak{P}))$ . Then  $\text{CT}(\mathfrak{F}) = \text{CT}(\mathfrak{P})$  by Lemma 6.3(a), so the  $\mathfrak{R}$ -contramodule  $\mathfrak{F}$  is flat. Following Lemma 6.2(b),  $\mathfrak{F}$  is also separated. According to Lemma 6.3(b), the adjunction morphism  $\mathfrak{P} \rightarrow \mathfrak{F}$  is surjective. Let  $\mathfrak{Q}$  denote its kernel. Following Lemma 6.10, we have a short exact sequence  $0 \rightarrow \text{CT}(\mathfrak{Q}) \rightarrow \text{CT}(\mathfrak{P}) \rightarrow \text{CT}(\mathfrak{F}) \rightarrow 0$  in  $\text{Fun}(\text{Discr-}\mathfrak{R})$ . Hence  $\text{CT}(\mathfrak{Q}) = 0$ , and by Lemma 6.14 we can conclude that  $\mathfrak{Q} = 0$ . Thus  $\mathfrak{P} = \mathfrak{F}$  and  $\mathfrak{P}$  is separated.  $\square$

**Lemma 6.16.** *Let  $(0 \rightarrow \mathfrak{Q}_i \rightarrow \mathfrak{P}_i \rightarrow \mathfrak{F}_i \rightarrow 0)_{i \in C}$  be a filtered diagram of short exact sequences in  $\mathfrak{R}\text{-Contra}$ . Assume that every  $\mathfrak{R}$ -contramodule  $\mathfrak{Q}_i$ ,  $\mathfrak{P}_i$ , and  $\mathfrak{F}_i$  is flat. Then the short sequence of colimits  $0 \rightarrow \text{colim}_i \mathfrak{Q}_i \rightarrow \text{colim}_i \mathfrak{P}_i \rightarrow \text{colim}_i \mathfrak{F}_i \rightarrow 0$  is exact in  $\mathfrak{R}\text{-Contra}$ .*

*Proof.* Following Corollary 6.15, the  $\mathfrak{R}$ -contramodules  $\mathfrak{F}_i$  are separated. Hence the short sequence of functors  $0 \rightarrow \mathrm{CT}(\mathfrak{Q}_i) \rightarrow \mathrm{CT}(\mathfrak{P}_i) \rightarrow \mathrm{CT}(\mathfrak{F}_i) \rightarrow 0$  is exact in  $\mathrm{Fun}(\mathrm{Discr}\text{-}\mathfrak{R})$  for every  $i$  by Lemma 6.10. Passing to the colimit, we obtain a short exact sequence of functors

$$0 \longrightarrow \mathrm{CT}(\mathrm{colim}_i \mathfrak{Q}_i) \longrightarrow \mathrm{CT}(\mathrm{colim}_i \mathfrak{P}_i) \longrightarrow \mathrm{CT}(\mathrm{colim}_i \mathfrak{F}_i) \longrightarrow 0.$$

By Lemma 6.6, we have a short exact sequence of left  $\mathfrak{R}$ -contramodules

$$0 \longrightarrow \mathrm{PL}(\mathrm{CT}(\mathrm{colim}_i \mathfrak{Q}_i)) \longrightarrow \mathrm{PL}(\mathrm{CT}(\mathrm{colim}_i \mathfrak{P}_i)) \longrightarrow \mathrm{PL}(\mathrm{CT}(\mathrm{colim}_i \mathfrak{F}_i)) \longrightarrow 0.$$

According to Lemma 5.6, the  $\mathfrak{R}$ -contramodules  $\mathrm{colim}_i \mathfrak{Q}_i$ ,  $\mathrm{colim}_i \mathfrak{P}_i$ , and  $\mathrm{colim}_i \mathfrak{F}_i$  are flat, and following Corollary 6.15 again, they are also separated. It remains to apply Lemma 6.3(c) in order to obtain the desired short exact sequence.  $\square$

**Proposition 6.17.** *Let  $(0 \rightarrow \mathfrak{Q}_i \rightarrow \mathfrak{P}_i \rightarrow \mathfrak{F}_i \rightarrow 0)_{i \in \mathcal{C}}$  be a filtered diagram of short exact sequences in  $\mathfrak{R}\text{-Contra}$ . Assume that every  $\mathfrak{R}$ -contramodule  $\mathfrak{F}_i$  is flat. Then the short sequence of colimits*

$$0 \longrightarrow \mathrm{colim}_i \mathfrak{Q}_i \longrightarrow \mathrm{colim}_i \mathfrak{P}_i \longrightarrow \mathrm{colim}_i \mathfrak{F}_i \longrightarrow 0$$

*is exact in  $\mathfrak{R}\text{-Contra}$ .*

*Proof.* For any small category  $\mathcal{C}$  and any cocomplete abelian category  $\mathcal{K}$  with enough projective objects, the category of functors  $\mathcal{K}^{\mathcal{C}}$  has enough projective objects. In fact, these are the direct summands of coproducts of functors of the form  $P^{(\mathcal{C}(A, -))}$ , where  $P$  is a projective object in  $\mathcal{K}$  and  $A$  an object in  $\mathcal{C}$ . Furthermore, for any projective object  $Q$  in  $\mathcal{K}^{\mathcal{C}}$  and any object  $C$  in  $\mathcal{C}$  the object  $Q(C)$  is projective in  $\mathcal{K}$ .

One constructs the left derived functors of  $\mathrm{colim}_{\mathcal{C}} : \mathcal{K}^{\mathcal{C}} \rightarrow \mathcal{K}$ ,  $n \geq 0$  by choosing a left projective resolution

$$Q_{\bullet} = (\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0) \longrightarrow F$$

in the abelian category of functors  $\mathcal{K}^{\mathcal{C}}$  for a given functor  $F : \mathcal{C} \rightarrow \mathcal{K}$ , computing the colimit  $\mathrm{colim}_{\mathcal{C}} Q_n$  of every functor  $Q_n : \mathcal{C} \rightarrow \mathcal{K}$ , and passing to the homology objects of the complex formed by such colimits in the category  $\mathcal{K}$ ,

$${}_n\mathrm{colim}_{\mathcal{C}}(F) = H_n(\mathrm{colim}_{\mathcal{C}} Q_{\bullet}).$$

Since the functor  $\mathrm{colim}_{\mathcal{C}} : \mathcal{K}^{\mathcal{C}} \rightarrow \mathcal{K}$  is right exact, one has  ${}_0\mathrm{colim}_{\mathcal{C}} F = \mathrm{colim}_{\mathcal{C}} F$ . This definition of a left derived functor provides for any short exact sequence of functors  $0 \rightarrow H \rightarrow G \rightarrow F \rightarrow 0$  in  $\mathcal{K}^{\mathcal{C}}$  a long exact sequence

$$\cdots \longrightarrow {}_1\mathrm{colim}_{\mathcal{C}} G \longrightarrow {}_1\mathrm{colim}_{\mathcal{C}} F \longrightarrow \mathrm{colim}_{\mathcal{C}} H \longrightarrow \mathrm{colim}_{\mathcal{C}} G \longrightarrow \mathrm{colim}_{\mathcal{C}} F \longrightarrow 0$$

in the category  $\mathcal{K}$ .

Now we specialize to the case of a filtered category  $\mathcal{C}$  of indices  $i$  and the category of contramodules  $\mathcal{K} = \mathfrak{R}\text{-Contra}$ . Let  $(\mathfrak{F}_i)_{i \in \mathcal{C}}$  be a diagram of flat  $\mathfrak{R}$ -contramodules. First we choose a diagram  $(\mathfrak{P}_i)_{i \in \mathcal{C}}$  in such a way that it is a projective object in  $\mathfrak{R}\text{-Contra}^{\mathcal{C}}$  and there is an epimorphism  $(\mathfrak{P}_i) \rightarrow (\mathfrak{F}_i)$  in  $\mathfrak{R}\text{-Contra}^{\mathcal{C}}$  with a kernel

$(\mathfrak{Q}_i)_{i \in \mathcal{C}}$ . Then, by the definition,  ${}_1\text{colim}_i \mathfrak{P}_i = 0$ . According to the above discussion, every  $\mathfrak{R}$ -contramodule  $\mathfrak{P}_i$  is projective; hence, by Lemma 6.9,  $\mathfrak{P}_i$  is flat. Due to Corollaries 6.8 and 6.15 we know that  $\mathfrak{Q}_i$  is flat for every  $i$ , too. Following Lemma 6.16, the short sequence  $0 \rightarrow \text{colim}_i \mathfrak{Q}_i \rightarrow \text{colim}_i \mathfrak{P}_i \rightarrow \text{colim}_i \mathfrak{F}_i \rightarrow 0$  is exact. Hence we can conclude from the long exact sequence

$$\cdots \longrightarrow {}_1\text{colim}_i \mathfrak{P}_i \longrightarrow {}_1\text{colim}_i \mathfrak{F}_i \longrightarrow \text{colim}_i \mathfrak{Q}_i \longrightarrow \text{colim}_i \mathfrak{P}_i \longrightarrow \text{colim}_i \mathfrak{F}_i \longrightarrow 0$$

that  ${}_1\text{colim}_i \mathfrak{F}_i = 0$ . Finally, for an arbitrary diagram  $(\mathfrak{P}_i) \in \mathfrak{R}\text{-Contra}^{\mathcal{C}}$  and a short exact sequence of diagrams  $(0 \rightarrow \mathfrak{Q}_i \rightarrow \mathfrak{P}_i \rightarrow \mathfrak{F}_i \rightarrow 0)_{i \in \mathcal{C}}$  the same long exact sequence of derived colimits shows that the short sequence  $0 \rightarrow \text{colim}_i \mathfrak{Q}_i \rightarrow \text{colim}_i \mathfrak{P}_i \rightarrow \text{colim}_i \mathfrak{F}_i \rightarrow 0$  is exact.  $\square$

**Remark 6.18.** Similarly one can show that  ${}_n\text{colim}_i \mathfrak{F}_i = 0$  for any filtered diagram of flat left  $\mathfrak{R}$ -contramodules  $(\mathfrak{F}_i)_{i \in \mathcal{C}}$  and all  $n \geq 1$ .

## 7. COTORSION THEORIES IN CONTRAMODULE CATEGORIES

We keep the assumptions of Section 6, i. e.,  $\mathfrak{R}$  is a complete and separated, associative and unital topological ring with a countable base of neighborhoods of zero formed by open right ideals. The following corollary summarizes the results of Section 6.

**Corollary 7.1.** *The full subcategory of flat  $\mathfrak{R}$ -contramodules in  $\mathfrak{R}\text{-Contra}$  is*

- (a) *closed under the kernels of epimorphisms;*
- (b) *closed under extensions;*
- (c) *strongly closed under directed colimits.*

*Proof.* Taking into account Corollary 6.15, part (a) is provided by Corollary 6.8 and part (b) by Corollary 6.13. Part (c) follows from Proposition 6.17 and Lemma 5.6.  $\square$

**Proposition 7.2.** *The full subcategory of flat  $\mathfrak{R}$ -contramodules is closed under  $\aleph_1$ -pure subobjects and  $\aleph_1$ -pure quotients in  $\mathfrak{R}\text{-Contra}$ .*

*Proof.* The argument is based on the construction of the derived functor  $\text{Ctrtor}_*^{\mathfrak{R}}(-, -)$  from the proof of Lemma 6.10. Let  $\mathfrak{P}$  be a flat left  $\mathfrak{R}$ -contramodule,  $\mathfrak{Q} \rightarrow \mathfrak{P}$  an  $\aleph_1$ -pure monomorphism, and  $\mathfrak{P} \rightarrow \mathfrak{F} = \mathfrak{P}/\mathfrak{Q}$  the corresponding  $\aleph_1$ -pure epimorphism. According to Proposition 5.5, the morphism of abelian groups  $N \odot_{\mathfrak{R}} \mathfrak{Q} \rightarrow N \odot_{\mathfrak{R}} \mathfrak{P}$  is a monomorphism for any discrete right  $\mathfrak{R}$ -module  $N$ . Following the proof of Lemma 6.10, we have  $\text{Ctrtor}_1^{\mathfrak{R}}(N, \mathfrak{P}) = 0$  for a flat  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ , thus we can conclude from the long exact sequence that  $\text{Ctrtor}_1^{\mathfrak{R}}(N, \mathfrak{F}) = 0$ .

Since projective left  $\mathfrak{R}$ -contramodules are flat by Lemma 6.9, taking the contratenor product of a short exact sequence of discrete right  $\mathfrak{R}$ -modules  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  with a complex of projective left  $\mathfrak{R}$ -contramodules  $\mathfrak{P}_\bullet$  produces a short exact sequence of complexes of abelian groups  $0 \rightarrow L \odot_{\mathfrak{R}} \mathfrak{P}_\bullet \rightarrow M \odot_{\mathfrak{R}} \mathfrak{P}_\bullet \rightarrow N \odot_{\mathfrak{R}} \mathfrak{P}_\bullet \rightarrow 0$ . Hence for any left  $\mathfrak{R}$ -contramodule  $\mathfrak{F}$  there is a long exact sequence

$$\cdots \longrightarrow \text{Ctrtor}_1^{\mathfrak{R}}(N, \mathfrak{F}) \longrightarrow L \odot_{\mathfrak{R}} \mathfrak{F} \longrightarrow M \odot_{\mathfrak{R}} \mathfrak{F} \longrightarrow N \odot_{\mathfrak{R}} \mathfrak{F} \longrightarrow 0,$$

and it follows that  $\mathfrak{F}$  is flat whenever one has  $\text{Ctrtor}_1^{\mathfrak{R}}(N, \mathfrak{F}) = 0$  for every discrete right  $\mathfrak{R}$ -module  $N$ . Finally, the  $\mathfrak{R}$ -contramodule  $\mathfrak{Q}$  is flat by Corollary 7.1(a).  $\square$

**Definition 7.3.** A left  $\mathfrak{R}$ -contramodule  $\mathfrak{G}$  is called *cotorsion* if  $\text{Ext}_{\mathfrak{R}\text{-Contra}}^1(\mathfrak{F}, \mathfrak{G}) = 0$  for any flat left  $\mathfrak{R}$ -contramodule  $\mathfrak{F}$ .

We will denote the class of all flat left  $\mathfrak{R}$ -contramodules by  $\mathfrak{R}\text{-Flat}$  and the class of all cotorsion left  $\mathfrak{R}$ -contramodules by  $\mathfrak{R}\text{-Cotors} \subseteq \mathfrak{R}\text{-Contra}$ . By the definition, the class  $\mathfrak{R}\text{-Cotors}$  is closed under extensions and products in  $\mathfrak{R}\text{-Contra}$ . Since the class of  $\mathfrak{R}\text{-Flat}$  contains projective  $\mathfrak{R}$ -contramodules and is closed under the kernels of epimorphisms (Lemma 6.9 and Corollary 7.1(a)), one has  $\text{Ext}_{\mathfrak{R}\text{-Contra}}^n(\mathfrak{F}, \mathfrak{G}) = 0$  for any flat  $\mathfrak{R}$ -contramodule  $\mathfrak{F}$ , any cotorsion  $\mathfrak{R}$ -contramodule  $\mathfrak{G}$ , and all  $n \geq 0$ . It follows that the class of cotorsion  $\mathfrak{R}$ -contramodules is closed under the cokernels of monomorphisms (see, e. g., [39, Lemma 6.17]).

**Example 7.4.** For any discrete right  $\mathfrak{R}$ -module  $N$  and left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ , there is a natural isomorphism of abelian groups

$$\text{Ext}_{\mathfrak{R}\text{-Contra}}^n(\mathfrak{P}, \mathbf{Ab}(N, \mathbb{Q}/\mathbb{Z})) = \mathbf{Ab}(\text{Ctrtor}_n^{\mathfrak{R}}(N, \mathfrak{P}), \mathbb{Q}/\mathbb{Z}), \quad n \geq 0,$$

as one can see by computing both the  $\text{Ext}$  and the  $\text{Ctrtor}$  in terms of the same left projective resolution of an  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$ . Consequently, the left  $\mathfrak{R}$ -contramodule  $\mathbf{Ab}(N, \mathbb{Q}/\mathbb{Z})$  is cotorsion for any discrete right  $\mathfrak{R}$ -module  $N$ .

**Corollary 7.5.** *The category of flat left  $\mathfrak{R}$ -contramodules is accessible.*

*Proof.* By Lemma 5.6,  $\mathfrak{R}\text{-Flat}$  is accessibly embedded into  $\mathfrak{R}\text{-Contra}$ , and by Proposition 7.2,  $\mathfrak{R}\text{-Flat}$  is closed under  $\aleph_1$ -pure subobjects. According to [1, Corollary 2.36], it follows that  $\mathfrak{R}\text{-Flat}$  is accessible.  $\square$

**Corollary 7.6.** *The class of flat  $\mathfrak{R}$ -contramodules is deconstructible in  $\mathfrak{R}\text{-Contra}$ .*

*Proof.* The class  $\mathfrak{R}\text{-Flat}$  is closed under transinitely iterated extensions in  $\mathfrak{R}\text{-Contra}$ , since it is closed under extensions and filtered colimits (Lemmas 7.1(b) and 5.6). It is closed under  $\aleph_1$ -pure subobjects and  $\aleph_1$ -pure quotients by Proposition 7.2. The class of all monomorphisms with flat cokernels is closed under filtered colimits in the category of all morphisms of  $\mathfrak{R}$ -contramodules  $\mathfrak{R}\text{-Contra}^{\rightarrow}$  by Proposition 6.17. Hence the desired assertion is provided by Lemma 4.14.  $\square$

**Corollary 7.7.** *Let  $(\mathcal{F}, \mathcal{C})$  be the cotorsion theory generated by a set of objects  $\mathcal{S}$  in  $\mathfrak{R}\text{-Contra}$ . Then any left  $\mathfrak{R}$ -contramodule has a special  $\mathcal{F}$ -precover, and any left  $\mathfrak{R}$ -contramodule that can be embedded as a subcontramodule into an  $\mathfrak{R}$ -contramodule from  $\mathcal{C}$  has a special  $\mathcal{C}$ -preenvelope.*

*Furthermore, an  $\mathfrak{R}$ -contramodule belongs to  $\mathcal{F}$  if and only if it is a direct summand of an  $\mathfrak{R}$ -contramodule from  $\text{filt}(\mathcal{S}')$ , where the set  $\mathcal{S}' = \mathcal{S} \cup \{\mathfrak{R}\}$  consists of the contramodules from  $\mathcal{S}$  and the free left  $\mathfrak{R}$ -contramodule  $\mathfrak{R}$ .*

*Proof.* Clearly, one has  $\mathcal{S}^\perp = \mathcal{S}'^\perp$ , so the cotorsion theory generated by  $\mathcal{S}$  in  $\mathfrak{R}\text{-Contra}$  coincides with the one generated by  $\mathcal{S}'$ . The class  $\text{filt}(\mathcal{S}')$  contains all the free left  $\mathfrak{R}$ -contramodules  $\mathfrak{R}[[X]]$ , and any left  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$  is a quotient contramodule of  $\mathfrak{R}[[X]]$  for a certain set  $X$ . All the assertions of the corollary now follow from the related assertions of Theorem 4.8.  $\square$

**Corollary 7.8.** *The pair of full subcategories  $(\mathfrak{R}\text{-Flat}, \mathfrak{R}\text{-Cotors})$  forms a complete cotorsion theory in  $\mathfrak{R}\text{-Contra}$ .*

*First proof.* By Corollary 7.6, there is a set of flat  $\mathfrak{R}$ -contramodules  $\mathcal{S}$  such that  $\mathfrak{R}\text{-Flat} = \text{filt}(\mathcal{S})$ . Replacing if needed  $\mathcal{S}$  by  $\mathcal{S}' = \mathcal{S} \cup \{\mathfrak{R}\}$ , we may assume that the set  $\mathcal{S}$  contains the free  $\mathfrak{R}$ -contramodule  $\mathfrak{R}$ . By Lemma 4.5, we have  $\mathfrak{R}\text{-Cotors} = \mathcal{S}^\perp$ . Denoting by  $(\mathcal{F}, \mathcal{C})$  the cotorsion theory generated by  $\mathcal{S}$  in  $\mathfrak{R}\text{-Contra}$ , we see that  $\mathcal{C} = \mathfrak{R}\text{-Cotors}$  and, following Corollary 7.7, the class  $\mathcal{F}$  consists precisely of the direct summands of objects from  $\text{filt}(\mathcal{S})$ . Hence  $\mathcal{F} = \mathfrak{R}\text{-Flat}$ . Now we know from Corollary 7.7 that any left  $\mathfrak{R}$ -contramodule has a special flat precover, and it only remains to prove that it has a special cotorsion preenvelope.

Any left  $\mathfrak{R}$ -contramodule  $\Omega$  is endowed with a natural  $\mathfrak{R}$ -contramodule morphism into a cotorsion  $\mathfrak{R}$ -contramodule

$$c_\Omega : \Omega \longrightarrow \prod_{\mathfrak{J}, f} \text{Ab}(\mathfrak{R}/\mathfrak{J}, \mathbb{Q}/\mathbb{Z}),$$

where  $\mathfrak{J}$  runs over all the open right ideals  $\mathfrak{J} \subseteq \mathfrak{R}$  and  $f \in \text{Ab}(\Omega/(\mathfrak{J} \triangleleft \Omega), \mathbb{Q}/\mathbb{Z})$ . The morphism  $c_\Omega$  assigns to any elements  $q \in \Omega$  and  $r \in \mathfrak{R}/\mathfrak{J}$ , and an abelian group homomorphism  $f$ , the element  $f(rq) \in \mathbb{Q}/\mathbb{Z}$ . The kernel of  $c_\Omega$  is precisely the  $\mathfrak{R}$ -subcontramodule  $\cap_{\mathfrak{J}} (\mathfrak{J} \triangleleft \Omega) \subseteq \Omega$ , i. e., the kernel of the adjunction morphism  $\Omega \rightarrow \text{PL}(\text{CT}(\Omega))$ . Applying Corollary 7.7 again, we can conclude that any separated left  $\mathfrak{R}$ -contramodule has a special cotorsion preenvelope.

Given an arbitrary left  $\mathfrak{R}$ -contramodule  $\Omega$ , let  $\mathfrak{F} \rightarrow \Omega$  be one of its special flat precovers. The  $\mathfrak{R}$ -contramodule  $\mathfrak{F}$  is separated by Corollary 6.15, hence it has a special cotorsion preenvelope  $\mathfrak{F} \rightarrow \mathfrak{G}$ . Let  $\mathfrak{K}$  be the kernel of the epimorphism of  $\mathfrak{R}$ -contramodules  $\mathfrak{F} \rightarrow \mathfrak{G}$ . Then the composition  $\mathfrak{K} \rightarrow \mathfrak{F} \rightarrow \mathfrak{G}$  is a monomorphism of cotorsion  $\mathfrak{R}$ -contramodules, so its cokernel  $\mathfrak{G}/\mathfrak{K}$  is also cotorsion. Now the natural morphism  $\Omega \rightarrow \mathfrak{G}/\mathfrak{K}$  is a special cotorsion preenvelope of  $\Omega$ . (This is the argument dual to [45, Lemma 3.1.3]; cf. [27, proof of Lemma 2 in Section 9.1].)  $\square$

*Second proof.* We have already seen in the first paragraph of the first proof that  $(\mathfrak{R}\text{-Flat}, \mathfrak{R}\text{-Cotors})$  is a cotorsion theory generated by a set of objects  $\mathcal{S}$  in  $\mathfrak{R}\text{-Contra}$  and that any left  $\mathfrak{R}$ -contramodule has a special flat precover. To show that it has a special cotorsion preenvelope, we apply Proposition 4.16. Indeed, the set  $\mathcal{S}$  is contained in the class of all flat left  $\mathfrak{R}$ -contramodules, which is closed under extensions and strongly closed under directed colimits in  $\mathfrak{R}\text{-Contra}$  by Corollary 7.1(b–c).  $\square$

*Third proof.* Deduce existence of special flat precovers of  $\mathfrak{R}$ -contramodules from existence of their flat covers, provable by the Bican–El Bashir approach as in the first



proof of the next Corollary 7.9. To prove existence of special cotorsion preenvelopes of subcontramodules of cotorsion contramodules, argue as in the proof of Salce's lemma [36, Lemma 2.3] (cf. the dual version in the second half of the proof of [17, Theorem 10]). Special cotorsion preenvelopes of arbitrary  $\mathfrak{R}$ -contramodules can be then obtained as in the second and third paragraphs of the first proof above.  $\square$

**Corollary 7.9.** *Any left  $\mathfrak{R}$ -contramodule has a flat cover.*

*First proof (Bican–El Bashir approach to flat covers).* Following Lemma 5.6, the full subcategory  $\mathfrak{R}\text{-Flat}$  is closed under directed colimits in  $\mathfrak{R}\text{-Contra}$ . Being an additive subcategory, it is consequently also closed under coproducts. By Corollary 7.5,  $\mathfrak{R}\text{-Flat}$  is accessible. Thus Corollary 2.8 allows to conclude that  $\mathfrak{R}\text{-Flat}$  is stably weakly coreflective in  $\mathfrak{R}\text{-Contra}$ .  $\square$

*Second proof (Eklof–Trlifaj approach to flat covers).* Since  $(\mathfrak{R}\text{-Flat}, \mathfrak{R}\text{-Cotors})$  is a complete cotorsion theory by Corollary 7.8 and  $\mathfrak{R}\text{-Flat}$  is closed under directed colimits by Lemma 5.6, we can conclude that  $\mathfrak{R}\text{-Flat}$  is stably weakly coreflective by Corollary 3.7.  $\square$

**Corollary 7.10.** *Any left  $\mathfrak{R}$ -contramodule has a cotorsion envelope.*

*Proof.* Since  $\mathfrak{R}\text{-Flat}$  is strongly closed under directed colimits in  $\mathfrak{R}\text{-Contra}$  by Corollary 7.1(c) and  $(\mathfrak{R}\text{-Flat}, \mathfrak{R}\text{-Cotors})$  is a complete cotorsion theory by Corollary 7.8,  $\mathfrak{R}\text{-Cotors}$  is stably weakly reflective in  $\mathfrak{R}\text{-Contra}$  by Corollary 3.10 or 4.18.  $\square$

**Corollary 7.11.** *Any cotorsion theory generated by a set of flat  $\mathfrak{R}$ -contramodules is complete in  $\mathfrak{R}\text{-Contra}$ .*

*First proof.* Let  $(\mathcal{F}, \mathcal{C})$  be the cotorsion theory generated by a set of flat  $\mathfrak{R}$ -contramodules  $\mathcal{S}$ . Following Corollary 7.7, any left  $\mathfrak{R}$ -contramodule has a special  $\mathcal{F}$ -precover, and any left  $\mathfrak{R}$ -contramodule embeddable into an object of  $\mathcal{C}$  has a special  $\mathcal{C}$ -preenvelope. Now we have  $\mathcal{S} \subseteq \mathfrak{R}\text{-Flat}$ , hence  $\mathcal{C} \supseteq \mathfrak{R}\text{-Contra}$ , and it remains to recall that any left  $\mathfrak{R}$ -contramodule is a subcontramodule of a cotorsion  $\mathfrak{R}$ -contramodule by Corollary 7.8.  $\square$

*Second proof.* To prove that any left  $\mathfrak{R}$ -contramodule has a special  $\mathcal{C}$ -preenvelope, notice that  $\mathcal{S} \subseteq \mathfrak{R}\text{-Flat}$  and the class  $\mathfrak{R}\text{-Flat}$  is strongly closed under directed colimits by Corollary 7.1(c), so Proposition 4.16 applies.  $\square$

**Examples 7.12.** (1) Let  $\mathfrak{R}$  be a complete and separated topological associative ring with a countable base of neighborhoods of zero  $B$  formed by open two-sided ideals (cf. Example 5.7(2)). Let  $\kappa$  be an uncountable regular cardinal. Since the fully faithful inclusion  $\mathfrak{R}/\mathfrak{I}\text{-Mod} \rightarrow \mathfrak{R}\text{-Contra}$  preserves  $\kappa$ -filtered colimits, the reduction functor  $\mathfrak{P} \mapsto \mathfrak{P}/(\mathfrak{I} \ltimes \mathfrak{P})$  takes  $\kappa$ -presentable  $\mathfrak{R}$ -contramodules to  $\kappa$ -presentable  $\mathfrak{R}/\mathfrak{I}$ -modules. Conversely, using Nakayama's lemma 6.14 one easily shows that any  $\mathfrak{R}$ -contramodule  $\mathfrak{P}$  whose reductions  $\mathfrak{P}/(\mathfrak{I} \ltimes \mathfrak{P})$  are  $\kappa$ -generated  $\mathfrak{R}/\mathfrak{I}$ -modules for all  $\mathfrak{I} \in B$  is  $\kappa$ -generated. Applying Lemma 6.7, one deduces the claim that any flat

$\mathfrak{R}$ -contramodule  $\mathfrak{F}$  whose reductions  $\mathfrak{F}/(\mathfrak{I} \times \mathfrak{F})$  are  $\kappa$ -presentable for all  $\mathfrak{I} \in B$  is  $\kappa$ -presentable. So a flat  $\mathfrak{R}$ -contramodule is  $\kappa$ -presentable if and only if its reductions by open ideals  $\mathfrak{I} \subseteq \mathfrak{R}$  are  $\kappa$ -presentable  $\mathfrak{R}/\mathfrak{I}$ -modules for all  $\mathfrak{I} \in B$ .

(2) Suppose that for every two-sided open ideal  $\mathfrak{I} \subseteq \mathfrak{R}$ ,  $\mathfrak{I} \in B$  we are given a deconstructible class of flat left  $\mathfrak{R}/\mathfrak{I}$ -modules  $\mathcal{F}_{\mathfrak{I}} \subseteq \mathfrak{R}/\mathfrak{I}\text{-Flat}$  such that for every pair of embedded ideals  $\mathfrak{I} \subseteq \mathfrak{I}' \subseteq \mathfrak{R}$ ,  $\mathfrak{I}, \mathfrak{I}' \in B$ , and any module  $F$  from  $\mathcal{F}_{\mathfrak{I}'}$  the  $\mathfrak{R}/\mathfrak{I}$ -module  $\mathfrak{R}/\mathfrak{I} \otimes_{\mathfrak{R}/\mathfrak{I}'} F$  belongs to  $\mathcal{F}_{\mathfrak{I}}$ . Then the class  $\mathcal{F} \subseteq \mathfrak{R}\text{-Flat}$  of all left  $\mathfrak{R}$ -contramodules  $\mathfrak{F}$  such that  $\mathfrak{F}/(\mathfrak{I} \times \mathfrak{F}) \in \mathcal{F}_{\mathfrak{I}}$  for every  $\mathfrak{I} \in B$  is deconstructible.

Specifically, let  $\kappa$  be an uncountable regular cardinal such that  $\mathcal{F}_{\mathfrak{I}} = \text{filt}(\mathcal{S}_{\mathfrak{I}})$  and  $\mathcal{S}_{\mathfrak{I}}$  is a set of  $\kappa$ -presentable  $\mathfrak{R}/\mathfrak{I}$ -modules for every  $\mathfrak{I} \in B$ . Let  $\mathcal{S}$  consist of all left  $\mathfrak{R}$ -contramodules  $\mathfrak{G} \in \mathcal{F}$  whose reductions  $\mathfrak{G}/(\mathfrak{I} \times \mathfrak{G})$  are  $\kappa$ -presentable for all  $\mathfrak{I} \in B$  (i. e., of all  $\kappa$ -presentable contramodules from  $\mathcal{F}$ ). We claim that  $\mathcal{F} = \text{filt}(\mathcal{S})$ .

Indeed, let  $\mathfrak{F}$  be a contramodule from  $\mathcal{F}$ , and let  $\mathfrak{I}_0 \supseteq \mathfrak{I}_1 \supseteq \mathfrak{I}_2 \supseteq \cdots \in B$  be a chain of ideals indexed by the nonnegative integers and cofinal in  $B$ . According to Hill's lemma [41], for every  $n \geq 0$  there exists a family of submodules  $\mathcal{M}_n$  in the  $R/\mathfrak{I}_n$ -module  $F_n = \mathfrak{F}/(\mathfrak{I}_n \times \mathfrak{F})$  closed under arbitrary sums and intersections in  $F_n$  and having the properties (H3) and (H4) from [41, Theorem 6]. These state that (H3) for any  $N$  and  $P \in \mathcal{M}_n$  with  $N \subseteq P$  one has  $P/N \in \mathcal{F}_{\mathfrak{I}_n}$ , and (H4) for any subset  $X \subseteq F_n$  of cardinality less than  $\kappa$  and  $N \in \mathcal{M}_n$  there exists  $P \in \mathcal{M}_n$  such that  $N \cup X \subseteq P$  and  $P/N$  is  $\kappa$ -presentable.

Let  $\lambda$  be the cardinality of  $\mathfrak{F}$ . For any nonnegative integers  $m \leq n$ , we have the reduction map  $f_{n,m} : F_n \rightarrow \mathfrak{R}/\mathfrak{I}_m \otimes_{\mathfrak{R}/\mathfrak{I}_n} F_n = F_m$ ; these maps form a commutative diagram and the reduction maps  $f_n : \mathfrak{F} \rightarrow F_n$  form a limit cone (since flat contramodules are separated). We construct smooth chains of submodules  $(F_{n,i})_{i \leq \lambda}$  in the  $R/\mathfrak{I}_n$ -modules  $F_n$  such that  $F_{n,i} \in \mathcal{M}_n$ ,  $F_{n,i+1}/F_{n,i}$  is  $\kappa$ -presentable, and  $F_{m,i} = f_{n,m}(F_{n,i})$  for  $m < n$ . Set  $F_{n,0} = 0$ .

On a successor step  $i+1$ , if  $F_{n,i} = F_n$  for all  $n$ , we set  $F_{n,i+1} = F_{n,i}$ . Otherwise, let  $m$  be the minimal nonnegative integer such that  $F_{m,i} \subsetneq F_m$ . Choose  $G_{m,1} \in \mathcal{M}_m$  such that  $F_{m,i} \subsetneq G_{m,1}$  and  $G_{m,1}/F_{m,i}$  is  $\kappa$ -presentable. Proceeding by induction in  $n \geq m$ , choose  $G_{n,1} \in \mathcal{M}_n$  such that  $G_{n,1}/F_{n,i}$  is  $\kappa$ -presentable and  $f_{n+1,n}(G_{n+1,1}) \supseteq G_{n,1}$ . Choose  $G_{m,2} \in \mathcal{M}_m$  such that  $f_{p,m}(G_{p,1}) \subseteq G_{m,2}$  for all  $p \geq m$  and  $G_{m,2}/F_{m,i}$  is  $\kappa$ -presentable. Proceeding by induction in  $k \geq 1$ , we choose submodules  $G_{n,k} \in \mathcal{M}_n$ ,  $n \geq m$  such that  $f_{n+1,n}(G_{n+1,k}) \supseteq G_{n,k}$ ,  $f_{p,n}(G_{p,k}) \subseteq G_{n,k+1}$  for all  $p \geq n \geq m$ , and  $G_{n,k}/F_{n,i}$  is  $\kappa$ -presentable. Now we can set  $F_{n,i+1} = F_{n,i}$  for  $n < m$  and  $F_{n,i+1} = \bigcup_k G_{n,k}$  for all  $n \geq m$ . Clearly  $F_{n,\lambda} = F_n$  for all  $n \geq 0$ .

Setting  $\mathfrak{F}_i = \lim_n F_{n,i}$ , we obtain a smooth chain of left  $\mathfrak{R}$ -contramodules  $(\mathfrak{F}_i \rightarrow \mathfrak{F}_j)_{i < j \leq \lambda}$  such that  $\mathfrak{F}_0 = 0$ ,  $\mathfrak{F}_{\lambda} = \mathfrak{F}$ , and the morphisms  $\mathfrak{F}_i \rightarrow \mathfrak{F}_{i+1}$  are  $\mathcal{S}$ -monomorphisms. We have shown that  $\mathcal{F} = \text{filt}(\mathcal{S})$ , hence according to Eklof's lemma 4.5 and Corollary 7.11 the cotorsion theory  $({}^{\perp}(\mathcal{F}^{\perp}), \mathcal{F}^{\perp})$  is complete in  $\mathfrak{R}\text{-Contra}$ . Following Corollary 7.7,  ${}^{\perp}(\mathcal{F}^{\perp})$  is the class of all direct summands of the contramodules from  $\text{filt}(\mathcal{S} \cup \{\mathfrak{R}\})$ .

(3) Now let  $\mathfrak{R}$  be a complete and separated topological commutative ring with a countable base of neighborhoods of zero. For any commutative ring  $R$ , we denote by  $(R\text{-Vfl}, R\text{-Ctadj})$  the cotorsion theory generated by the set of  $R$ -modules  $R[s^{-1}]$ ,  $s \in R$ . Modules from  $R\text{-Vfl}$  are called *very flat* and modules from  $R\text{-Ctadj}$  are called *contraadjusted*; so  $R\text{-Vfl} \subseteq R\text{-Flat}$  and  $R\text{-Ctadj} \supseteq R\text{-Cotors}$  (see [29, Section 1.1]).

An  $\mathfrak{R}$ -contramodule  $\mathfrak{F}$  is called *very flat* (cf. [29, Sections C.3 and D.4]) if the  $\mathfrak{R}/\mathfrak{I}$ -module  $\mathfrak{F}/(\mathfrak{I} \ltimes \mathfrak{F})$  is very flat for every open ideal  $\mathfrak{I} \subseteq \mathfrak{R}$ . An  $\mathfrak{R}$ -contramodule  $\mathfrak{G}$  is *contraadjusted* if  $\text{Ext}_{\mathfrak{R}\text{-Contra}}^1(\mathfrak{F}, \mathfrak{G}) = 0$  for any very flat  $\mathfrak{R}$ -contramodule  $\mathfrak{F}$ . The class of very flat  $\mathfrak{R}$ -contramodules is denoted by  $\mathfrak{R}\text{-Vfl}$  and the class of contraadjusted  $\mathfrak{R}$ -contramodules by  $\mathfrak{R}\text{-Ctadj}$ .

Let  $\mathcal{S}_R$  denote the (representative) set of all  $\aleph_1$ -presentable very flat  $R$ -modules. Following [41, Lemma 9], we have  $R\text{-Vfl} = \text{filt}(\mathcal{S}_R)$ . Hence, according to (2),  $\mathfrak{R}\text{-Vfl} = \text{filt}(\mathcal{S}_{\mathfrak{R}})$ , where  $\mathcal{S}_{\mathfrak{R}}$  consists of all  $\mathfrak{R}$ -contramodules  $\mathfrak{F}$  such that  $\mathfrak{F}/(\mathfrak{I} \ltimes \mathfrak{F}) \in \mathcal{S}_{\mathfrak{R}/\mathfrak{I}}$  for all open ideals  $\mathfrak{I} \subseteq \mathfrak{R}$ . Since the class  $\mathfrak{R}\text{-Vfl}$  contains the free  $\mathfrak{R}$ -contramodules and is closed under direct summands, we can conclude that  $(\mathfrak{R}\text{-Vfl}, \mathfrak{R}\text{-Ctadj})$  is a complete cotorsion theory in the category  $\mathfrak{R}\text{-Contra}$ .

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